PLANE GEOMETRY

AMERICAN SCHOOL OF CORRESPONDENCE
CHICAGO, ILLINOIS

40

Digitized by the Internet Archive
in 2023 with funding from
Kahle/Austin Foundation

https://archive.org/details/planegeometry0000edwa

PLANE GEOMETRY

INSTRUCTION PAPER

PREPARED BY

EDWARD B. WAITE
DEAN AND HEAD, CONSULTING DEPARTMENT
AMERICAN SCHOOL OF CORRESPONDENCE
AMERICAN SOCIETY OF MECHANICAL ENGINEERS

AMERICAN SCHOOL OF CORRESPONDENCE
CHICAGO ILLINOIS
U.S.A.

COPYRIGHT, 1911, 1914, BY
AMERICAN SCHOOL OF CORRESPONDENCE

COPYRIGHTED IN GREAT BRITAIN
ALL RIGHTS RESERVED

PLANE GEOMETRY.

INTRODUCTORY DEFINITIONS.

1. **Mathematics** is the science of quantity.

2. **Quantity** is that which can be measured; as distance, time, weight.

3. **Geometry** is that branch of mathematics which treats of the properties of extension.

4. **Extension** has one or more of the three dimensions, length, breadth, or thickness.

5. A **Point** has position, but not magnitude.

6. A **Line** has length, without breadth or thickness.

7. A **Straight Line** is one whose direction is the same throughout.

A straight line has two directions exactly opposite, of which either may be assumed as its direction.

(NOTE. The word line used alone in this book, means a straight line.)

8. **Corollary**. One theorem easily derived from another.

9. A **Curved Line** is one whose direction is constantly changing.

10. A **Surface** has length and breadth, but no thickness.

11. A **Plane** is such a surface that a straight line joining any two of its points is wholly in the surface.

12. A **Solid** has length, breadth, and thickness.

13. The boundaries of solids are surfaces; of surfaces, lines; the ends of lines are points.

14. A **Theorem** is a truth which requires to be proved.

15. A **Problem** is a question proposed for solution.

16. A **Proposition** is a general name for either a theorem or a problem.

16 a. One theorem is the **Converse** of another when the conclusion of the first is made the hypothesis of the second, and the hypothesis of the first is made the conclusion of the second.

17. A **Corollary** is an inference from a proposition or statement.

18. A **Scholium** is a remark appended to a proposition.

19. A theorem consists of two parts, the **Hypothesis**, that which is given, and the **Conclusion**, that which is to be proved.

20. An **Axiom** is a self-evident truth.

AXIOMS.

1. If equals be added to equals, the sums are equal.
2. If equals are subtracted from equals, the remainders are equal.
3. If equals are multiplied by equals, the products are equal.
4. If equals are divided by equals, the quotients are equal.
5. Like powers and like roots of equals are equal.
6. The whole of a magnitude is greater than any of its parts.
7. The whole of a magnitude is equal to the sum of all its parts.
8. Magnitudes respectively equal to the same magnitude are equal to each other.
9. A straight line is the shortest distance between two points.

ABBREVIATIONS.

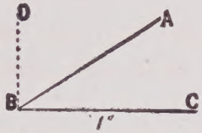
The following list of abbreviations will be found useful:

+, plus.
—, minus.
×, multiplied by.
=, equals,
∟, right angle.
⊥, perpendicular.
⊥s, perpendiculars.
∥, parallel.
∥s, parallels.
○, circle.
Ⓢ, circles.
>, is greater than.
<, is less than.
∴, therefore.
∠, angle.
∠s, angles.
△, triangle.
△s, triangles.
▱, parallelogram.
▱s, parallelograms.
⌒, arc.
≎, equivalent to.

ANGLES, LINES, POLYGONS.

GENERAL PRINCIPLES AND DEFINITIONS.

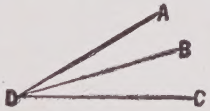
21. When two straight lines meet at a point, the difference in their direction is an *angle*. The point of meeting is called the *vertex*, and the lines the *sides*, of the angle. Thus in the angle formed by the lines A B and B C, B is the vertex, and A B and B C are the sides.



22. In speaking of an angle at a given vertex, it can be designated by the letter at that vertex, if, as here, there is but one angle at that vertex. But when two or more angles have a common vertex, then it is necessary, to avoid ambiguity, to name also the letters at the extremities of the sides, placing the letter at the vertex in the middle. Thus we say here "angle B"; but if D B be drawn, we should have to say "angle A B C," or "angle C B A" to name the same angle before designated.

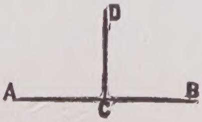
23. The size of an angle depends solely upon the amount of divergence of its sides, not at all upon their length.

24. When two angles have the same vertex and a side common to both, they are called *adjacent* angles; as angle A D B, and angle B D C.

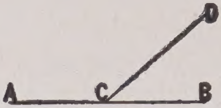


The side D B is common to both.

25. If a straight line meets another so as to make the adjacent angles equal, each of these angles will be a right angle; and the two lines will be perpendicular to each other. Thus angle A C D and angle D C B, being equal, are right angles, and A B and D C are perpendicular to each other.



26. An **Acute Angle** is less than a right angle; as angle B C D.



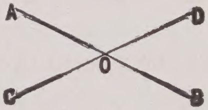
27. An obtuse angle is greater than a right angle; as angle A C D.

Acute and obtuse angles are called *oblique* angles.

28. If the sum of two angles is equal to a right angle, or 90°, each is called the *complement* of the other; if their sum is equal to two right angles, 180°, they are called the *supplements* of each other.

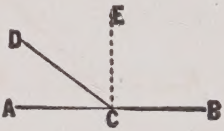
It is evident that complements of equal angles are equal to each other; and the same is true of their supplements.

29. Two angles are called *vertical* or *opposite*, when the sides of one are the prolongations of the sides of the other, as angle A O C and angle B O D.



THEOREM I.

30. *If one straight line meets another straight line, the adjacent angles thus formed, are equal to two right angles.*



Let D C meet the straight line A B at the point C.

To prove that $ACD + DCB =$ two right angles.

At C, erect the perpendicular C E; then it is evident that

$$ACD + DCE = ACE = \text{one right angle.}$$

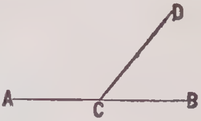
And

$$BCD - DCE = BCE = \text{one right angle}$$

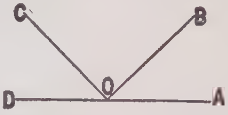
Adding we have

$$ACD + BCD = ACE + BCE = \text{two right angles.}$$

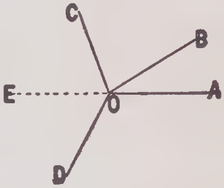
31. *Corollary* 1. If only two angles are formed, each is the supplement of the other. For by the Theorem,



A C D + B C D = 2 right angles.
∴ A C D = 2 right angles — D C B,
or D C B = 2 right angles — A C D.



32. *Corollary* 2. The sum of all the angles A O B, B O C, C O D, formed on the same side of a straight line at a given point = two right angles. For, by the Theorem, A O B + B O D = two right angles, and this sum is equal to the sum of the angles A O B, B O C and C O D.

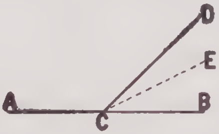


33. *Corollary* 3. The sum of all the angles A O B, B O C, C O D and D O A formed about a point is equal to four right angles. For if the line A O be produced to E, by the preceding corollary, the sum of the angles A O B, B O C and C O E = two right angles, and the same is true of the sum of the angles A O D and D O E.

Hence, the sum of the angles A O B, B O C, C O D, and D O A = four right angles.

THEOREM II.

34. *When the sum of two adjacent angles is equal to two right angles, their exterior sides form a straight line.*



Let the sum of the adjacent angles A C D and B C D be equal to two right angles.

To prove that A C B is a straight line.

If A C B is not a straight line, let C E be in the same straight line with A C.

By Corollary 1, angle E C D is the supplement of angle A C D.

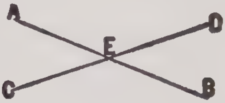
But by hypothesis angle B C D is the supplement of A C D.

Therefore angle B C D = angle E C D, which is impossible unless C E coincides with C B.

Hence C B is in the same straight line with A C.

THEOREM III.

35. *If two straight lines cut each other, the vertical angles are equal.*



Let the straight lines A B and C D intersect at E.

To prove that angle A E C = angle B E D.

By § 30, angle A E C is the supplement of angle A E D.

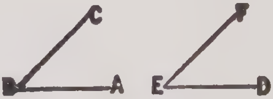
Angle B E D is also the supplement of angle A E D; and

∴ angle A E C = angle B E D. (By § 28.)

In like manner it may be proved that angle A E D = angle B E C.

THEOREM IV.

36. *Two angles whose sides have the same or opposite directions are equal.*

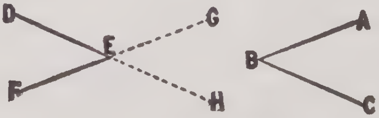


1st, Let B A and B C, including angle B, have respectively the same directions as E D and E F, including the angle E;

To prove angle B = angle E.

Since B A has same direction as E D, and B C has same direction as E F, the difference of direction of B A and B C must be the same as the difference of direction of E D and E F.

∴ angle B = angle E.

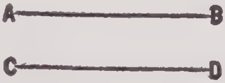


2nd, Let B A and B C, including angle B, have respectively opposite directions to E F and E D, including angle E;

To prove angle B = angle E.
Produce D E and F E so as to form angle G E H;
Then, angle G E H = angle D E F (Theorem III),
And, angle G E H = angle A B C (By hypothesis).
∴ angle B = angle E.

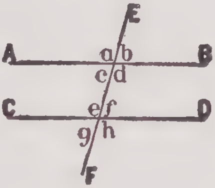
PARALLEL LINES.

37. *Definition.* Two straight lines are called parallel when they lie in the same plane and cannot meet however far they may be produced, as A B and C D.



38. *Axiom.* Two lines parallel to a third are parallel to each other.

39. *Definitions.* If two straight lines A B and C D are cut by a third line E F, the eight angles thus formed are named as follows:



The angles *c*, *d*, *e*, *f* are called *interior* angles.

The angles *a*, *b*, *g*, *h* are called *exterior* angles.

The angles *c* and *f* or *d* and *e* are called *alternate-interior* angles.

The angles *a* and *h* or *b* and *g* are called *alternate-exterior* angles.

The angles *a* and *e*, *b* and *f*, *c* and *g*, or *d* and *h* are called *corresponding* angles.

THEOREM V.

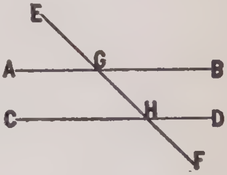
40. *If two parallels are cut by a third straight line:*

1*st.* *All the corresponding angles are equal.*

2*nd.* *The alternate-interior angles are equal.*

3*rd.* *The interior (as also the exterior) angles on the same side (of the intersecting line* E F*), are supplements of each other.*

Let E F cut the parallels A B and C D. To prove:



1st. The corresponding angles E G A and G H C, or E G B and G H D, are equal, since their sides have the same direction. (Theorem IV).

2nd. The alternate interior angles A G H and G H D, or B G H and G H C, are equal, because their sides have opposite directions. (Theorem IV).

3rd. The internal angles on the same side, A G H and G H C, or B G H and G H D, are supplements of each other; for angle A G H is the supplement of angle A G E (corollary of Theorem I), which has been proved equal to angle G H C. In the same way it may be proved that angles B G H and G H D are supplements of each other.

THEOREM VI.

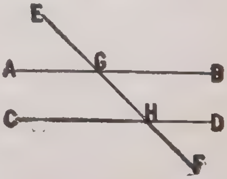
41. *Conversely if a straight line cut two other straight lines in the same plane, these two lines are parallel.*

I. If the corresponding angles are equal.

II. If the alternate interior angles are equal.

III. If the interior, as also the exterior, angles on the same side (of the intersecting line) are supplements of each other.

Let E F cut the two lines A B and C D so as to make angle E G B = angle G H D, or angle A G H = angle G H D, or angles B G H and G H D supplements of each other.



To prove that A B is parallel to C D:

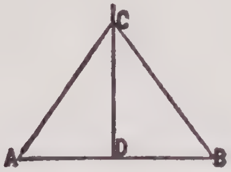
If through the point G a line is drawn parallel to C D, it will make the corresponding angles equal, and the alternate interior angles equal, and the interior angles, as also the exterior, on the same side (of the intersecting line) supplements of each other, by (Theorem V); therefore it must coincide with A B; that is, A B is parallel to C D.

THEOREM VII.

42. *If a perpendicular is erected at the middle point of a straight line.*

I. Any point in the perpendicular is equally distant from the extremities of the line.

II. Any point without the perpendicular is unequally distant from the extremities of the line.



I. Let C D be perpendicular to A B at its middle point D, and let C be any point in C D.

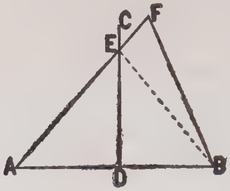
To prove that C is equally distant from A and B:

Join A C and B C.

Let the figure B D C be superposed upon A D C by folding it over about D C as an axis.

Since angle B D C = angle A D C (adjacent angles, formed by a perpendicular meeting another line), and B D = A D (by hypothesis), the point B will fall at A and B C will coincide with A C.

Therefore B C = A C, and C is equally distant from A and B.



II. Let C D be perpendicular to A B at its middle point D, and let F be any point without C D.

To prove that F is unequally distant from A and B.

Join A F and B F; let A F intersect C D at E, and draw B E.

Since the straight line B F is shorter than the broken line B E F (a straight line being the shortest distance between two points),

B F is less than B E + E F.

Now we have already proved that B E = A E, (the first part of this theorem.)

Hence B F is less than A E + E F, or B F is less than A F.

Therefore F is unequally distant from A and B.

43. *Corollary* 1. When the figure B D E is superposed upon A D E, the angles E B D and B E D coincide with the angles E A D and A E D, respectively: that is,
angle E A D = angle E B D, and angle A E D = angle B E D.

Hence, *if straight lines are drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point, they make equal angles with the line and with the perpendicular.*

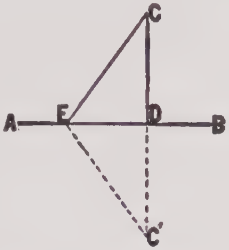
44. *Corollary* II. If a series of points, all of which satisfy a given condition, lie in a certain line, that line is called the *locus* of the points. For example, all the points which satisfy the condition of being equally distant from the extremities of a straight line lie in the perpendicular erected at its middle point.

In other words, *the perpendicular at the middle point of a straight line is the locus of points which are equally distant from the extremities of the line.*

45. *Scholium.* We know by *axiom* that a straight line is determined by any two of its points. It therefore follows that *two points equally distant from the extremities of a line determine its perpendicular at its middle point.*

THEOREM VIII.

46. *The perpendicular is the shortest line which can be drawn from a point to a straight line.*



Let C D be the perpendicular from C to A B, and C E any oblique line.

To prove that C D is less than C E.

Produce C D to C′, making C′ D = C D, and join E C′.

Then since E D is perpendicular to C C′ at its middle point, C E = C′ E, (Theorem VII.)

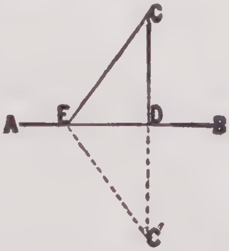
But the straight line C C′ being shorter than the broken line C E C′, we have

C D + C′ D is less than C E + C′ E

Therefore, 2 C D is less than 2 C E, or C D is less than C E.

THEOREM IX.

47. *From a given point outside a straight line, but one perpendicular can be drawn to that line.*



Let C be the given point outside the line A B, and draw C D perpendicular to A B.

To prove that C D is the only perpendicular which can be drawn.

If possible, let C E be another perpendicular from C to A B.

Produce C D to C′, making C′ D = C D, and then join C′ E.

Then since E D is perpendicular to C C′ at its middle point, angle C′ E D = angle C E D (§ 43.)

But C E D is a right angle (by hypothesis.)

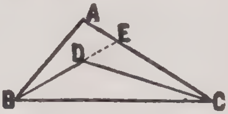
Hence C′ E D is also a right angle, and the sum of the adjacent angles C E D and C′ E D is equal to two right angles, and therefore C E C′ is a straight line (Theorem II.)

Now the latter is impossible, since but one straight line can be drawn between two given points (*Axiom.*)

Hence C E cannot be perpendicular to A B, and C D is the only perpendicular which can be drawn.

THEOREM X.

48. *If two lines are drawn from a point to the extremities of a straight line, their sum is greater than the sum of two other lines similarly drawn, but enveloped by them.*



Let A B and A C be drawn from the point A to the extremities of the line B C, and let D B and D C be two lines similarly drawn but enveloped by A B and A C.

To prove that A B + A C is greater than D B + D C:

Produce B D to meet A C at E.

Then since the broken line B A E is greater than the straight line B E, the line B A C is greater than B E C.

And since the broken line D E C is greater than the straight line D C, the line B E C is greater than B D C.

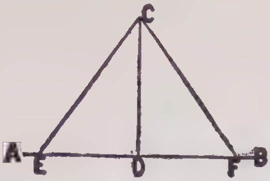
Therefore, A B + A C is greater than D B + D C.

THEOREM XI.

49. *If oblique lines are drawn from a point to a straight line.*

I. Two oblique lines cutting off equal distances from the foot of the perpendicular are equal.

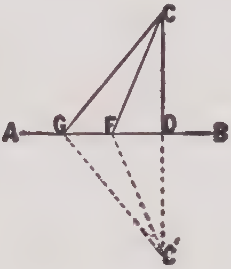
II. Of two oblique lines cutting off unequal distances from the foot of the perpendicular, the more remote is the greater.



I. Let the oblique lines C E and C F meet the line A B at equal distances from the foot of the perpendicular C D.

To prove that C E = C F.

Since C D is perpendicular to E F at its middle point D, C E = C F (Theorem VII.)



II. Let the oblique lines C F and C G meet the line A B at unequal distances from the foot of the perpendicular C D, and let C G be the more remote.

To prove that C G is greater than C F:

Produce C D to C′, making C′ D = C D, and then join C′ F and C′ G.

Now C F = C′ F, and C G = C′ G. (Theorem VII).

But the broken line C G C′ is greater than the broken line C F C′, which is enveloped by it. (Theorem X).

Hence C G + C′ G is greater than C F + C′ F.

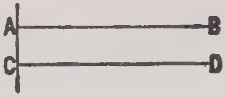
Therefore 2 C G is greater than 2 C F, or C G is greater than C F.

50. *Corollary I.* Conversely two equal oblique lines from a point to a straight line, cut off equal distances from the foot of the perpendicular.

51. *Corollary II.* Only two equal straight lines can be drawn from the same point to the same straight line.

THEOREM XII.

52. *Two straight lines perpendicular to the same straight line are parallel.*



Let the lines A B and C D be perpendicular to A C.

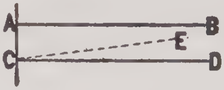
To prove that they are parallel:

If they are not parallel, they will meet if sufficiently produced (*definition*), and we should then have two perpendiculars from the same point to the same straight line, which is impossible. (Theorem IX).

Therefore, they cannot meet and are parallel.

THEOREM XIII.

53. *A straight line perpendicular to one of two parallels is perpendicular to the other.*



Let A B and C D be parallel lines, and let A C be perpendicular to A B.

To prove that A C is also perpendicular to C D:

Let C E be perpendicular to A C.

Then C E is parallel to A B. (Theorem XII).

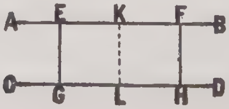
But C D is parallel to A B (*by hypothesis*).

Hence C E must coincide with C D, since but one straight line can be drawn through the point C parallel to A B (*by axiom*).

Therefore A C is perpendicular to C D.

THEOREM XIV.

54. *Two parallel lines are everywhere equally distant.*



Let A B and C D be two parallel lines, and let E and F be any two points on A B.

To prove that E and F are equally distant from C D:

Let K be the middle point of E F, and draw E G, K L, and F H perpendicular to A B and C D. (Theorem XIII).

Let the portion of the figure to the right of K L be folded over about K L as an axis until it comes into the plane of the figure on the left.

Since angle F K L = angle E K L (right angles by construction), and F K = E K (by construction), the line F K will fall upon E K and the point F will fall upon E. And since, in the same way, angle K L D = angle K L C, the line D L will fall upon C L.

Therefore the line F H, which is perpendicular to D L will coincide with E G, which is perpendicular to C L (Theorem IX), and the point H will fall at G.

Hence E G = F H and the points E and F are equally distant from C D.

PLANE FIGURES.

DEFINITIONS.

55. **A Plane Figure** is a portion of a plane bounded by lines, either curved or straight.

If the bounding lines are straight the figure is a **polygon,** and the sum of the bounding lines is the **perimeter.**

56. An **Equilateral Polygon** is one whose sides are equal each to each.

57. An **Equiangular Polygon** is one whose angles are equal each to each.

58. Polygons whose sides or angles are respectively equal are called *mutually* equilateral or mutually equiangular.

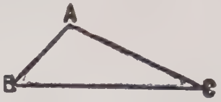
Two equal sides, or two equal angles, one in each polygon, similarly situated, are called *homologous* sides or angles.

59. Polygons are spoken of as **equal** only when, being applied to each other, they exactly coincide.

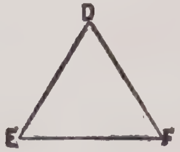
60. Of Polygons, the simplest has three sides, and is called a *triangle;* one of four sides is a *quadrilateral;* one of five, a *pentagon:* one of six, a *hexagon;* one of eight, an *octagon;* one of ten a *decagon.*

TRIANGLES.

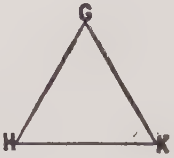
61. When a triangle has no two of its sides equal, it is termed *scalene;* as A B C.



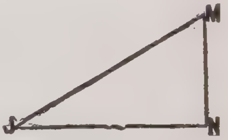
62. When a triangle has two of its sides equal, it is termed *isosceles*; as D E F.



63. When a triangle has all its sides equal, it is termed *equilateral;* as G H K.

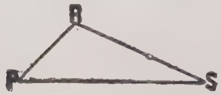


64. When one of the angles of a triangle is a right angle, the triangle is termed a *right triangle,* as J M N.

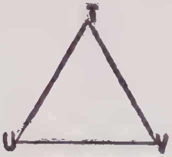


The side J M opposite the right angle is termed the *hypothenuse.*

65. When one of the angles of a triangle is an obtuse angle, the triangle is termed an *obtuse-angled triangle;* as P R S.



66. When all the angles of a triangle are acute angles, the triangle is termed an *acute-angled triangle;* as T U V.



67. The side upon which any polygon is supposed to be resting is usually called its *base;* but in an isosceles triangle, as D E F above, in which DE = D F, the third side E F is always considered the base.

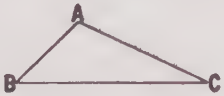
When any side has been taken as *base*, the opposite angle is called the *vertical angle*, and its vertex is called the vertex of the triangle.

The *altitude* of a triangle is the perpendicular drawn from the vertex to the base.

68. Since a straight line is the shortest distance between two points, it follows that *either side of a triangle is less than the sum of the other two.*

THEOREM XV.

69. *Either side of a triangle is greater than the difference of the other two.*



Let A B C be any triangle.

To prove that any side as A B, is greater than the difference of the other two:

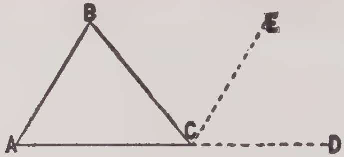
Since a straight line is the shortest distance between two points, A B + A C is greater than B C.

Subtracting A C from both members of the inequality, A B is greater than B C — A C.

$$\begin{aligned} AB + AC &> BC \\ AC &= AC \\ AB &> BC - AC \end{aligned}$$

THEOREM XVI.

70. *The sum of the angles of any triangle is equal to two right angles.*



Let A B C be any triangle.

To prove that the sum of angles A, B, and C is equal to two right angles:

Produce A C to D and draw C E parallel to A B; then angle D C E = angle A (corresponding angles); angle B C E = angle B (alternate angles). Hence angles D C E + B C E + B C A = A + B + B C A, but angles D C E + B C E + B C A = two right angles. (Theorem I.)

Therefore angles A + B + B C A = two right angles.

71. *Corollary I.* If any side of a triangle is produced, the exterior angle thus formed is equal to the sum of the two opposite interior angles.

73. *Corollary II.* If two triangles have two angles of the one respectively equal to two angles of the other, the remaining angles are equal.

74. *Corollary III.* A triangle cannot have two right or two obtuse angles; for in either case the sum of the three angles in the triangle would exceed two right angles.

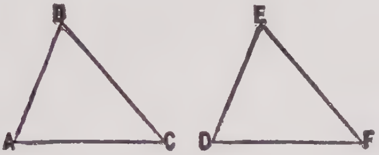
75. *Corollary IV.* In a right triangle the sum of the two acute angles is equal to a right angle.

76. *Corollary V.* If two angles of a triangle are known, the third can be formed by subtracting their sum from two right angles.

76 a. Corollary VI. In a right triangle the two acute angles are complimentary.

THEOREM XVII.

77. *If two triangles have two sides and the included angle of the one equal respectively to two sides and the included angle of the other, the two triangles are equal in all respects.*



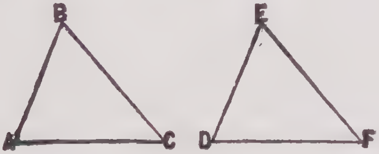
In the triangles A B C and D E F, let the side A B = D E, A C = D F and the angle A = angle D.

To prove triangle A B C = triangle D E F:

Superpose A B C upon D E F, so that A B coincides with its equal D E; then as angle A = angle D, and as A C = D F, A C will take the direction of D F and will coincide with it. Therefore the extremities of the third lines B C and E F respectively coinciding, the lines themselves must do so, and the two triangles are proved to be equal in all respects.

THEOREM XVIII.

78. *If two triangles have a side and the two adjacent angles of the one equal respectively to a side and the two adjacent angles of the other, the two triangles are equal.*



In the triangles A B C and D E F, let angle A = angle D, angle C = angle F, and the side A C = D F.

To prove that triangle A B C is equal in all respects to triangle D E F:

Place the side A C on its equal D F, with the point A on the point D, and the point C on the point F, as A C = D F (by hyp.); then since angle A = angle D, A B will take the direction of

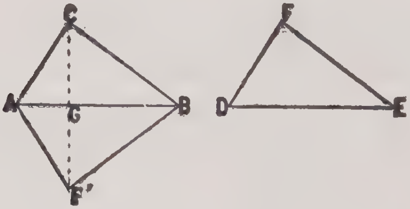
D E, and since angle C = angle F, C B will take the direction of F E; and the point B falling at once in each of the lines D E and F E must be at their point of intersection E; therefore the two triangles coincide and are equal.

79. *Corollary I.* Two right triangles are equal when the hypothenuse and an acute angle of one are respectively equal to the hypothenuse and an acute angle of the other; for the other acute angles are equal (Theorem XVI, Corollary II, and the triangles are then equal by Theorem XVIII).

Corollary II. Two right triangles are equal when a side and an acute angle of one are respectively equal to a side and an acute angle of the other.

THEOREM XIX.

80. *Two triangles are equal when the three sides of the one are equal respectively to the three sides of the other.*



In the triangles A B C and D E F, let the side A B = D E, B C = E F, and C A = F D.

To prove that the triangles are equal in all respects:

Place the triangle D E F in the position A B F′, so that D E shall coincide with its equal A B, the vertex F falling at F′ on the opposite side of A B from C.

Draw C F′ intersecting A B at G.

Since by hypothesis A C = A F′ and B C = B F′, the the points A and B are equally distant from C and F′.

Hence the line A B is perpendicular to C F′ at its middle point G (Theorem VII).

Let now the triangle A B C be folded over upon A B as axis, until it falls into the plane of A B F′.

Since angle A G C = angle A G F′, and C G = F′ G, the point C will fall at F′.

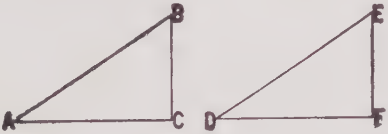
Therefore the triangles coincide throughout and are equal.

Corollary. Since the triangles coincide throughout,

angle A = angle D,
angle B = angle E,
angle C = angle F.

THEOREM XX.

81. *Two right triangles having the hypothenuse and a side of one respectively equal to the hypothenuse and a side of the other, are equal in all respects.*



In the right triangles A B C and D E F, let the hypothenuse A B be equal to D E, and the side B C equal to E F.

To prove that the triangles are equal:

Superpose the triangle A B C upon D E F so that the side B C shall coincide with its equal E F, the point B falling at E and C at F.

Then since angle C = angle F, the side A C will fall upon D F.

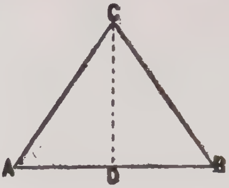
But the equal oblique lines A B and D E cut off upon D F equal distances from the foot of the perpendicular E F (Article 50).

That is, the point A will fall at D.

Therefore the triangles coincide throughout and are equal.

THEOREM XXI.

82. *In an isosceles triangle the angles opposite the equal sides are equal.*



Let A B C be an isosceles triangle in which A C and B C are the equal sides.

To prove that angle A = angle B:

Draw C D perpendicular to A B.

There in the right triangles A D C and B D C, the side C D is common and A C = B C by hypothesis.

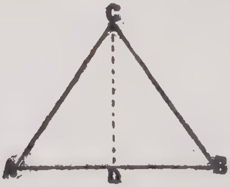
Therefore by Theorem XX, the triangles are equal in all respects, and angle A = angle B.

83. *Corollary I.* From the equality of the triangles A C D and B C D, A D = D B, and the angle A C D = angle B C D; that is, the line bisecting the angle opposite the base of an isosceles triangle bisects the base at right angles, and also bisects the triangle. Also the line drawn from the vertex perpendicular to the base of an isosceles triangle, bisects the base, the vertical angle, and the triangle.

84. *Corollary II.* An equilateral triangle is equiangular.

THEOREM XXII.

85. Conversely, *if two angles of a triangle are equal, the sides opposite them are equal, and the triangle is isosceles.*



In the triangle A B C let the angles A and B be equal.

To prove that A C = B C:

Draw C D perpendicular to A B.

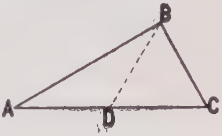
Then in the right triangles A C D and B C D the side C D is common, and by hypothesis angle A = angle B.

Therefore the triangles are equal in all respects (Theorem XVIII, Corollary II), and A C = B C.

86. *Corollary.* An equiangular triangle is also equilateral.

THEOREM XXIII.

87. *The greater side of a triangle is opposite the greater angle; and conversely, the greater angle is opposite the greater side.*



In the triangle A B C let angle B be greater than C;

To prove that A C is greater than A B:

At the point B make the angle C B D equal to the angle C;

Then D B = D C (Theorem XXII), and A C = A D + D C = A D + D B.

But (by axiom 9) A D + D B is greater than A B.

Therefore A C is greater than A B.

Conversely. Let A C be greater than A B.

NOTE.—In a figure for this proof D would fall nearer C.

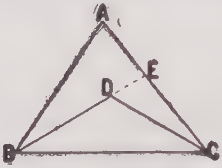
To prove that angle A B C is greater than angle C:

Cut off A D = A B and join B D; then as A D = A B, the angle A B D = angle A D B (Theorem XXI), and angle A D B is greater than angle C (Theorem XVI, Corollary I.)

Therefore angle A B D is greater than angle C; but angle A B C is greater than angle A B D; therefore angle A B C is greater than angle C.

THEOREM XXIV.

88. *If two straight lines are drawn from a point within a triangle to the extremities of any side, the angle included by them is greater than the angle included by the other two sides.*



Let D B and D C be drawn from any point D within the triangle A B C to the extremities of any side, as B C.

To prove that angle B D C is greater than angle A:

Produce B D to meet A C at E.

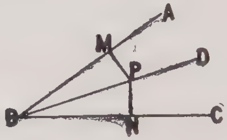
Then the angle B D C, being equal to the sum of the opposite interior angles D E C and D C E (Theorem XVI, Corollary I), is greater than angle D E C.

And the angle D E C, being equal to the sum of the opposite interior angles A and A B E, is greater than angle A.

Therefore angle B D C is greater than angle A.

THEOREM XXV.

89. *Any point in the bisector of an angle is equally distant from the sides of the angle.*



Let B D be the bisector of an angle, A B C, and let P be any point in B D.

To prove that P is equally distant from A B and B C:

Draw P M and P N perpendicular to A B and B C respectively.

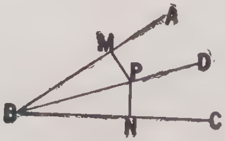
Then in the right triangles P B M and P B N, the side P B is common, and by hypothesis angle P B M = angle P B N.

Therefore the triangles are equal (Theorem XVIII, Corollary I), and P M = P N.

Hence P is equally distant from A B and B C.

THEOREM XXVI.

90. Conversely, *every point which is within an angle, and equally distant from its sides, lies in the bisector of the angle.*



Let P be equally distant from the sides of the angle A B C; join P and B by the line P B.

To prove that P B is the bisector of the angle A B C:

Draw P M and P N, perpendicular respectively to A B, and B C.

Then in the right triangles P B M and P B N, the side P B is common, and by hypothesis P M = P N.

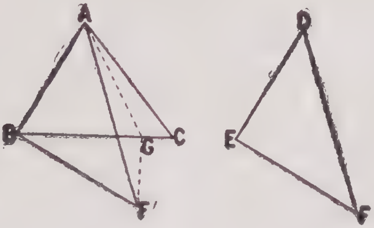
Therefore the triangles are equal, and angle P B M = angle P B N. (Theorem XX).

Hence P B bisects angle A B C.

91. *Scholium.* The bisector of the angle is the *locus* of points which are within the angle and equally distant from its sides.

THEOREM XXVII.

92. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*



In the triangles A B C and D E F, let the side A B be equal to D E, and A C to D F, and let the included angle B A C be greater than angle E D F.

To prove that B C is greater than E F:

Place the triangle D E F in the position A B F′, the side D E coinciding with its equal A B, and the vertex F falling at F′.

Draw A G bisecting the angle C A F′, and join F′ G.

Then in the triangles A C G and A F′ G, the side A G is common, A C = A F′ by hypothesis, and angle C A G = angle F′ A G by construction.

Therefore the triangles are equal in all respects (Theorem XVII), and F′ G = C G.

But by axiom (a straight line being the shortest distance between two points), B G + F′ G is greater than B F′.

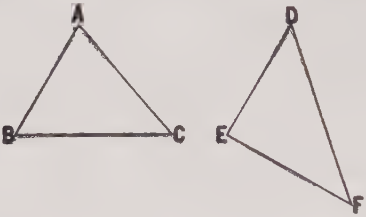
Substituting for F′ G its equal C G,

B G + C G is greater than B F′

That is, B C is greater than E F.

THEOREM XXVIII.

93. Conversely, *if two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.*



In the triangles A B C and D E F, let the side A B be equal to D E, and A C to D F, and let B C be greater than E F.

To prove that angle A is greater than angle D:

If angle A were equal to angle D, the triangles would be equal (Theorem XVII), and B C would be equal to E F.

And if angle A were less than angle D, B C would be less than E F (Theorem XXVII).

Both of these conclusions are contrary to the hypothesis;

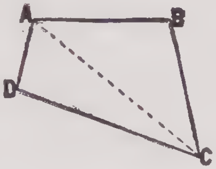
Hence angle A is greater than angle D.

94. *Scholium.* The method of proof employed in Theorem XXVIII, is known as the "Reductis ad absurdum," or "indirect" method. The truth of the theorem is demonstrated by supposing it to be false, and showing that the hypothesis leads to a false conclusion.

QUADRILATERALS.

DEFINITIONS.

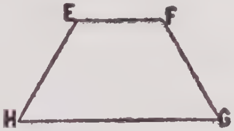
95. A **Quadrilateral** is a plane figure bounded by four straight lines, as A B C D.



96. A **diagonal** of a quadrilateral is a straight line joining two opposite vertices, as A C.

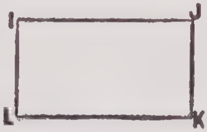
97. A **Trapezium** is a quadrilateral which has no two of its sides parallel; as A B C D.

98. A **Trapezoid** is a quadrilateral which has only two of its sides parallel; as E F G H.

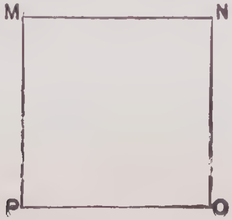


99. A **Parallelogram** is a quadrilateral whose opposite sides are parallel, as any rectangle or rhombus.

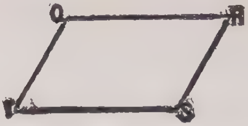
100. A **Rectangle** is a right-angled parallelogram, as I J K L.



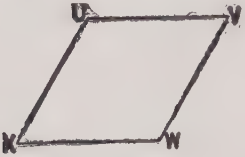
101. A **Square** is an equilateral rectangle, as M N O P.



102. A **Rhomboid** is an oblique-angled parallelogram, as Q R S T.



103. A **Rhombus** is an equilateral parallelogram, as U V W X.

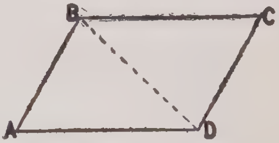


104. The **bases** of a trapezoid are its parallel sides; the **altitude** is the perpendicular distance between them.

105. The bases of a parallelogram are the side upon which it is supposed to stand and its parallel side; the altitude is the perpendicular distance between them.

THEOREM XXIX.

106. *The opposite sides and angles of a parallelogram are equal to each other.*



Let A B C D be a parallelogram;

To prove that A B = D C, B C = A D, and the angles A and B respectively equal C and D:

Draw the diagonal B D. In the triangles A B D and B C D, the side B D is common. As B C and A D are parallel, the alternate interior angles C B D, and B D A are equal. (Theorem V.) And in the same way angle A B D = angle B D C.

Therefore the two triangles A B D and B D C are equal (Theorem XVIII;) and the sides opposite the equal angles are equal, i.e. A B = D C, and B C = A D; also the angle A = angle C and the angle

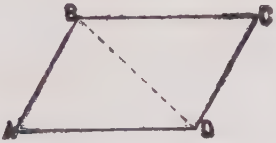
A B C = angle A B D + D B C = B D C + B D A = A D C.

107. *Corollary I.* The diagonal divides a parallelogram into two equal triangles.

108. *Corollary II.* Parallels included between parallels are equal.

THEOREM XXX.

109. *If two sides of a quadrilateral are equal and parallel the figure is a parallelogram.*



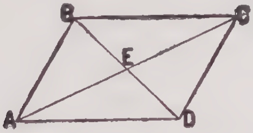
Let A B C D be a quadrilateral having B C equal and parallel to A D;

To prove that A B C D is a parallelogram:

Draw the diagonal B D. As B C is parallel to A D, the alternate interior angles C B D and B D A are equal (Theorem V), therefore the two triangles C B D and B D A are equal (Theorem XVII); and the alternate interior angles A B D and B D C are equal. Therefore A B is parallel to D C (Theorem VI), and A B C D is a parallelogram.

THEOREM XXXI.

110. *The diagonals of a parallelogram bisect each other.*



Let the diagonals of the parallelogram A B C D intersect at E.

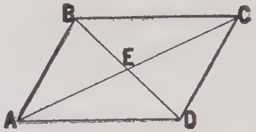
To prove that A E = E C and B E = E D:

In the triangles A E D and B E C, we have A D = B C, (Theorem XXIX), also since the parallels A D and B C are cut by A C, angle E A D = angle E C B. And since the parallels A D and B C are cut by B D, angle E D A = angle E B C.

Therefore the triangles are equal (Theorem XVIII), and A E = E C, and B E = E D (homologous sides of equal triangles).

THEOREM XXXII.

111. Conversely, *if the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*



Let the diagonals of the quadrilateral A B C D bisect each other at E.

To prove that the figure is a parallelogram.

In the triangles A E D and B E C, A E = E C, and D E = E B by hypothesis.

Also angle A E D = angle B E C (vertical angles).

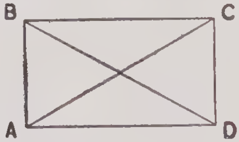
Therefore the triangles are equal (Theorem XVII) and A D = B C, (homologous.)

Similarly the triangles A E B and C E D are equal, and A B = C D, (homologous.)

Hence the opposite sides of the quadrilateral are equal, and the figure is a parallelogram, (Converse of Theorem XXIX, easily proved, by finding that the two triangles formed by the diagonal of such a figure are equal by Theorem XIX; three sides of one equalling three sides of the other.)

THEOREM XXXIII.

112. *The diagonals of a rectangle are equal.*



Let A B C D be a rectangle.

To prove that A C = B D.

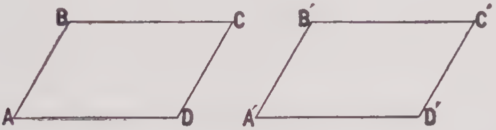
In the right triangles A B C and B C D, the side B C is common, and A B = C D (Theorem XXIX.)

Therefore the triangles are equal (Theorem XVII), and A C = B D (homologous).

113. *Corollary.* The diagonals of a square are equal, and bisect each other at right angles; for the square is at once a rectangle and a rhombus.

THEOREM XXXIV.

114. *Two parallelograms are equal when two adjacent sides and the included angle of one are equal respectively to two adjacent sides and the included angle of the other.*



In the parallelograms A B C D and A′ B′ C′ D′, let A B = A′ B′ and A D = A′ D′, and let the included angle A = angle A′.

To prove that the parallelograms are equal.

Since angle A = angle A′, the parallelogram A B C D may be superposed upon A′ B′ C′ D′ in such a way that the side A B shall coincide with its equal A′ B′, and A D with A′ D′.

Then since B C is parallel to A D and B′ C′ to A′ D′, the side B C will fall upon B′ C′, and the point C will fall somewhere in B′ C′.

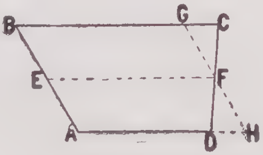
And since D C is parallel to A B and D′ C′ to A′ B′, the side D C will fall upon D′ C′ and the point C will fall somewhere in D′ C′.

Therefore the point C, being at the same time in B′ C′ and D′ C′, must fall at their point of intersection C′.

Hence the parallelograms coincide throughout and are equal.

THEOREM XXXV.

115. *The line joining the middle points of the two non-parallel sides of a trapezoid is parallel to the two parallel sides, and equal to half their sum.*



Let E F join the middle points of the non-parallel sides A B and C D, of the trapezoid A B C D.

To prove:

I. That E F is parallel to A D and B C.

Through F draw G H parallel to B A, meeting A D produced, at H. The angles G F C and D F H are equal (vertical angles); also the angles G C F and F D H (alternate interior angles); and the side C F is equal to F D (hypothesis). Therefore the triangles G C F and F D H are equal (Theorem XVIII), and $GF = FH = \frac{1}{2} GH$. But as A B G H is a parallelogram, $GH = BA$ (Theorem XXIX); therefore $FH = \frac{1}{2} BA = AE$.

Therefore A E F H is a parallelogram (Theorem XXX), and E F is parallel to A D, and therefore also to B C.

II. That $EF = \frac{1}{2}(AD + BC.)$

Since A E F H and E B G F are parallelograms,

$$EF = AH = AD + DH$$

and also

$$EF = BG = BC - GC$$

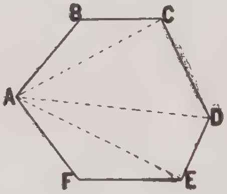
Now, as the two triangles G F C and D F H are equal, $GC = DH$; therefore if we add the two equations:

$$2\,EF = AD + BC$$

$$EF = \frac{1}{2}(AD + BC)$$

THEOREM XXXVI.

116. *The sum of the interior angles of a polygon is equal to twice as many right angles as it has sides minus two.*



Let A B C D E F oe the given polygon.

To prove that the sum of the interior angles A, B, C, D, E, and F, is equal to twice as many right angles as the figure has sides minus two.

If from any vertex as A, diagonals A C, A D, A E, are drawn, the polygon will be divided into as many triangles as the figure has sides minus two; and since we know that the sum of the angles of each triangle is equal to two right angles, the sum of the angles of all the triangles, that is, the sum of the interior angles of the polygon, is equal to twice as many right angles as the polygon has sides minus two.

RATIO AND PROPORTION.

DEFINITIONS.

(NOTE. It is necessary to understand the elementary principles of ratio and proportion before entering upon the theorems that are to follow. It is therefore introduced here.)

117. Ratio is the relation of one quantity to another of the same kind; or it is the quotient obtained by dividing one quantity by another of the same kind.

Ratio is indicated by writing the two quantities thus:

$a : b$, or $\frac{a}{b}$; read a is to b, or a divided by b.

118. The **Terms** of a ratio are the quantities compared, whether simple or compound.

The first term of a ratio is called the *antecedent*, the other the *consequent*; the two terms together are called a *couplet*.

119. An **Inverse** or **Reciprocal Ratio** of any two quantities is the ratio of their *reciprocals*. Thus the *direct* ratio of a to b is $a : b$, that is $\frac{a}{b}$; the *inverse* ratio of a to b is $\frac{1}{a} : \frac{1}{b}$, that is $\frac{1}{a} \div \frac{1}{b} = \frac{b}{a}$, or $b : a$.

120. Proportion is an equality of ratios.

Four quantities are in proportion when the ratio of the first to the second equals the ratio of the third to the fourth.

The equality of two ratios is indicated by the sign of equality (=), or by four dots (: :).

Thus $a : b = c : d$, or $a : b : : c : d$, or $\frac{a}{b} = \frac{c}{d}$; read a to b equals c to d, a is to b as c is to d, or a divided by b equals c divided by d.

121. In a proportion, the antecedents and consequents of the two ratios are respectively the *antecedents* and *consequents* of the proposition. The first and fourth terms are called the *extremes*, the second and third the *means*.

122. In a proportion in which the means are equal, e. g., $a : b = b : c$, either mean is called a *mean proportional* between the other two; and the third, a *third proportional* to the first and second.

123. A proportion is transformed by **Alternation** when antecedent is compared with antecedent, and consequent with consequent.

124. A proportion is transformed by **Inversion** when the antecedents are made consequents and the consequents antecedents.

125. A proportion is transformed by **Composition** when in each couplet the sum of the antecedent and consequent is compared with the antecedent or with the consequent.

126. A proportion is transformed by **Division** when in each couplet the difference of the antecedent and consequent is compared with the antecedent or with the consequent.

127. **Axiom.** Two ratios respectively equal to a third are equal to each other.

The following are the theorems of ratio and proportion:

THEOREM I.

128. *In a proportion the product of the extremes is equal to the product of the means.*

Let $a : b = c : d$

That is $\frac{a}{b} = \frac{c}{d}$

Clearing of fractions, $ad = bc$

129. *Scholium.* A proportion is merely an equation; and when we make the product of the extremes equal to the product of the means, we simply clear of fractions.

THEOREM II.

130. *If (conversely to Theorem I) the product of the two quantities be equal to the product of two others, one pair may be made the extremes, and another the means of a proportion.*

Let $ad = bc$

Dividing by bd, $\frac{ad}{bd} = \frac{bc}{bd}$ or $\frac{a}{b} = \frac{c}{d}$

That is, $a : b = c : d$

Similarly we may prove that:

$$a : c = b : d$$
$$b : d = a : c$$
$$c : d = a : b, \text{ etc.}$$

THEOREM III.

131. *In any proportion the terms are in proportion by* **Alternation**; *that is, the first term is to the third as the second is to the fourth.*

Let $a : b = c : d$

By (128) $ad = bc$

By (130) $a : c = b : d$

THEOREM IV.

132. *In any proportion the terms are in proportion by* **Inversion**; *that is, the second term is to the first as the fourth is to the third.*

Let $a : b = c : d$

By (128) $ad = bc$

By (130) $b : a = d : c$

THEOREM V.

133. *In any proportion the terms are in proportion by* **Composition**; *that is, the sum of the first two terms is to the first term, as the sum of the last two terms is to the third term.*

Let $a : b = c : d$

That is $\frac{a}{b} = \frac{c}{d}$

Adding 1 to each member,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

or, $\frac{a + b}{b} = \frac{c + d}{d}$

That is, $a + b : b = c + d : d$

THEOREM VI.

134. *In any proportion the terms are in proportion by* **Division**; *that is, the difference of the first two terms is to the first term, as the difference of the last two terms is to the third term.*

Let $a : b = c : d$

That is, $\frac{a}{b} = \frac{c}{d}$

Subtracting 1 from each member, $\frac{a}{b} - 1 = \frac{c}{d} - 1$

or, $\frac{a-b}{b} = \frac{c-d}{d}$

That is, $a - b : b = c - d : d$

135. *Corollary.* From (133) and (134), and by means of (131) and (127):

If $a : b = c : d$

Then $a + b : a - b = c + d : c - d$

THEOREM VII.

136. *Equimultiples of two quantities have the same ratio as the quantities themselves.*

For $\frac{a}{b} = \frac{ma}{mb}$

That is $a : b = ma : mb$

137. *Corollary.* It follows that either couplet of a proposition may be multiplied or divided by any quantity, and the resulting quantities will be in proportion. And since by (131), if $a : b = ma : mb$, $a : ma = b : mb$ or $ma : a = mb : b$, it follows that both consequents, or both antecedents, may be multiplied or divided by any quantity, and the resulting quantities will be in proportion.

THEOREM VIII.

138. *In any proportion, like powers or like roots of the terms are in proportion.*

Let $a : b = c : d$

That is $\frac{a}{b} = \frac{c}{d}$

Hence, $\frac{a^n}{b^n} = \frac{c^n}{d^n}$

That is, $a^n : b^n = c^n : d^n$

Since n may be either fractional or integral, the theorem is proved.

THEOREM IX.

139. *If any number of quantities are proportional, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let $a:b = c:d = e:f$

Now, $ab = ab$ (A)

And by (128), $ad = bc$ (B)

And also, $af = be$ (C)

Adding (A), (B), (C),

$$a\,(b + d + f) = b\,(a + c + e)$$

Hence by (130), $a:b = a + c + e : b + d + f$

THEOREM X.

140. If there are two sets of quantities in proportion, their products, or quotients, term by term, will be in proportion.

Let $a:b = c:d$

And $e:f = g:h$

By (128), $ad = bc$ (A)

And $eh = fg$ (B)

Multiplying (A) by (B), $adeh = bcfg$ (C)

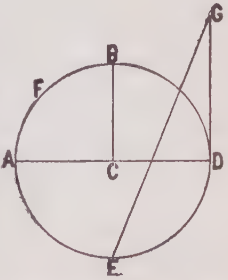
Dividing (A) by (B), $\frac{ad}{eh} = \frac{bc}{fg}$ (D)

From (C) by (130), $ae:bf = cg:dh$

And from (D), $\frac{a}{e}:\frac{b}{f} = \frac{c}{g}:\frac{d}{h}$

THE CIRCLE.

DEFINITIONS.



141. A **Circle** is a plane figure bounded by a curved line called the *circumference*, every point of which is equally distant from a point within called the *centre*; as A B D E.

142. A **Radius** of a circle is a straight line joining the centre with the circumference, as B C.

143. A **Diameter** of a circle is a straight line drawn through the centre, terminating at both ends in the circumference, as A D.

144. *Corollary.* The radii of a circle, or of equal circles, are evidently equal; also the diameters are equal, and each diameter is equal to twice the radius.

145. An **Arc** is any part of the circumference; as A F B. When an arc equals one-half the circumference, it is called a *Semi-circumference;* when it equals one-fourth a circumference, it is called a *quadrant.*

146. A **Chord** is the straight line joining the ends of an arc, as for example, a straight line joining A and B. The arc is said to be *subtended* by its chord.

147. A **Segment** of a circle is the portion included between an arc and its chord.

148. A **Sector** is the part of a circle included between two radii and the intercepted arc; as the space B C D.

149. A **Tangent** (in geometry) is a straight line which touches, but does not, though produced, cut the circumference, as G D. The point at which the tangent touches the circumference is called the *point of contact*, or *point of tangency.*

150. A **Secant** is a straight line which, if sufficiently produced, will cut the circumference in two points, as G E.

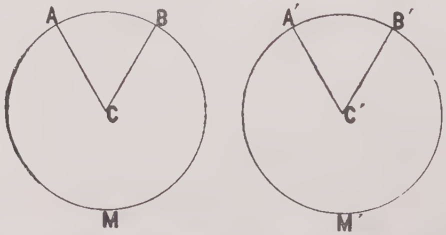
151. Two circles are *concentric* when they have the same centre.

152. A *central* angle is one whose vertex is at the centre, the sides being radii, as angle A C B.

An *inscribed* angle is one whose vertex is on the circumference, the sides being chords.

THEOREM XXXVII.

153. *In equal circles, or in the same circle, equal central angles intercept equal arcs.*



In the equal circles A M B and A′ M′ B′, let C and C′ be equal central angles.

To prove that arc A B = arc A′ B′:

Superpose the sector A B C upon A′ B′ C′ in such a way that the angle C shall coincide with its equal C′.

Then since the radii of the circles are equal, the point A will fall at A′ and B at B′.

Therefore the arc A B will coincide with the arc A′ B′, since all points of each are equally distant from the centre.

Hence, arc A B = arc A′ B′.

THEOREM XXXVIII.

154. Conversely, *in equal circles, or in the same circle, equal arcs are intercepted by equal central angles.*

In the equal circles A M B and A′ M′ B′, let the arc A B = arc A′ B′.

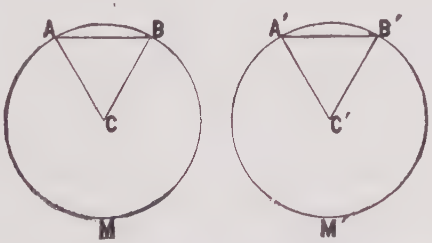
To prove that angle C = angle C′:

Since arc A B = arc A′ B′, the circle A B M may be suprepposed upon A′ B′ M′ in such a way that the point A shall fall at A′ and B at B′, the centre C falling at C′.

Then the radii A C and B C will coincide with A′ C′ and B′ C′ respectively.

THEOREM XXXIX.

155. *In equal circles, or in the same circle, equal chords subtend equal arcs.*



In the equal circles A M B and A′ M′ B′, let the chord A B be equal to chord A′ B′.

To prove that arc A B = arc A′ B′:

Let C and C′ be the centres of the circles, and draw A C, B C, A′ C′, and B′ C′.

Then in the triangles A B C and A′ B′ C′, A B = A′ B′, by hypothesis.

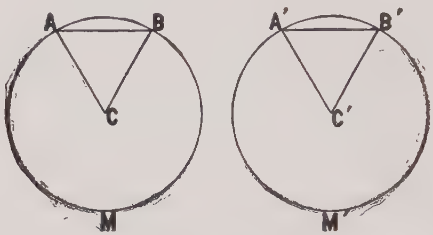
And since the radii of circles are equal, A C = A′ C′, and B C = B′ C′.

Hence, the triangles are equal, (Theorem XIX), and angle C = angle C′.

Hence, arc A B = arc A′ B′ (Theorem XXXVII).

THEOREM XL.

156. Conversely, *in equal circles, or in the same circle, equal arcs are subtended by equal chords.*



In the equal circles A M B and A′ M′ B′, let the arc A B = arc A′ B′.

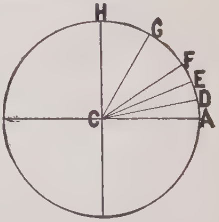
To prove that chord A B = chord A′ B′:

Let C and C′ be the centres of the circles, join A C, B C, A′ C′, and B′ C′.

Then in the triangles A B C and A′ B′ C′, A C = A′ C′, and B C = B′ C′. And since arc A B = arc A′ B′, angle C = angle C′, (Theorem XXXVIII), and chord A B = chord A′ B′.

THEOREM XLI.

157. *Central angles vary as their corresponding arcs.*

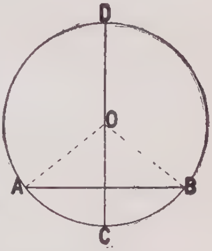


Let A C D, D C E, E C F be equal central angles at C; then the arcs A D, D E, E F are equal (Theorem XXXVII); then the angle A C E is double the angle A C D, and the arc A E is double the arc A D; and the angle A C F three times the angle A C D, and the arc A F three times the arc A D; and if the angle A C G is *m* times the angle A C D, the arc A G is *m* times the arc A D; that is, the angle varies as the arc, or the arc as the angle.

158. *Corollary.* As central angles vary as their arcs, or arcs vary as their corresponding angles, either of these quantities may be assumed as the measure of the other. The measure of an angle is, then, *the arc included between its sides and described from its vertex as a center.*

THEOREM XLII.

159. *The diameter perpendicular to a chord bisects the chord and its subtended arc.*



In the circle A B D, let the diameter C D be perpendicular to the chord A B.

To prove that it bisects A B and its subtended arcs:

Let O be the centre of the circle; join O A, and O B.

Then since O A = O B, the triangle O A B is isosceles; and the line C D, passing through the vertex perpendicular to the base, bisects the base and also the vertical angle, (by Theorem XXI, Corollary I).

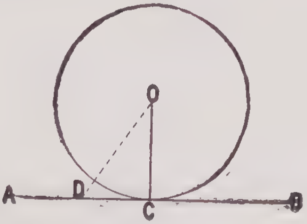
Hence, angle A O C = angle B O C, and arc A C = arc B C, (Theorem XXXVII).

Subtracting the equal arcs A C and B C from the semi-circumferences C A D and C B D, we have arc A D = arc B D.

Therefore the diameter bisects the chord A B and its subtended arcs A C B and A D B.

THEOREM XLIII.

160. *A straight line perpendicular to a radius at its extremity, is tangent to the circumference.*



Let A B be perpendicular to the radius O C, at its extremity C.

To prove that it is tangent to the circumference:

Since the perpendicular O C is the shortest line which can be drawn from O to A B (Theorem VIII), any other line, as O D, is greater than O C, and hence the point D lies without the circle.

That is, every point of A B except C lies without the circle, and A B is therefore tangent to the circumference (article 149).

161. *Corollary.* Conversely, *a tangent to the circumference is perpendicular to the radius drawn to the point of contact.*

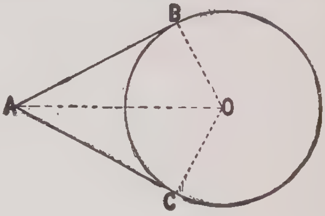
For if A B is tangent to the circumference at C, every point of A B except C lies without the circle.

Hence, O C is the shortest line which can be drawn from O to A B, and is therefore perpendicular to A B.

THEOREM XLIV.

162. *The two tangents to a circumference from an outside point are equal.*

Let A B and A C be the tangents from the point A to the circumference whose centre is O.



To prove that A B = A C.

Join O A, and draw the radii O B and O C. Since O B is perpendicular to A B, and O C to A C, (by Theorem XLIII, Corollary I), A B O and A C O are right angles.

Then in right triangles O A B and O A C, O A is common, and O B = O C.

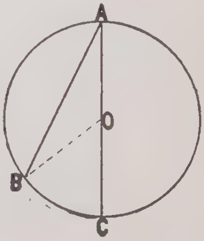
Hence the triangles are equal (by Theorem XX), and A B = A C.

THEOREM XLV.

163. *An inscribed angle is measured by one-half its intercepted arc.*

Case I. *When one side of the angle is a diameter.*

Let A C be a diameter of the circle A B C.



To prove that the inscribed angle A is measured by one-half arc B C.

Draw the radius O B. Then (by Theorem XVI Corollary I), angle B O C = angle A + angle B.

But since A O B is an isosceles triangle (O A and O B being radii of circle A B C) angle A = angle B.

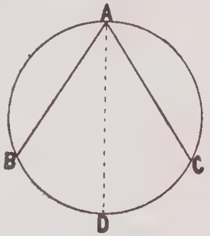
Hence, angle B O C = 2 angle A, or angle A = one-half angle B O C.

Now angle B O C is measured by arc B C (Theorem XLI, Corollary I.)

Therefore angle A is measured by one-half arc B C.

Case II. *When the centre is within the angle.*

In the circle A B C let the centre lie within the angle B A C.



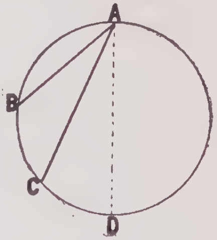
To prove that angle B A C is measured by one-half arc B C.

Draw the diameter A D. Then angle B A C = B A D + C A D.

But by Case I, angle B A D is measured by one-half arc B D, and angle C A D is measured by one-half arc C D.

Hence angle B A C is measured by one-half (arc B D + arc C D), or by one-half arc B C.

Case III. *When the centre lies without the angles.*



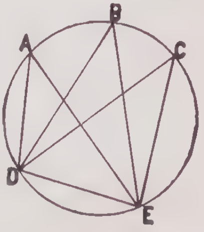
In the circle A B C let the centre lie without the angle B A C.

To prove that angle B A C is measured by one-half arc B C.

Draw the diameter A D. Then angle B A C = angle B A D — angle C A D.

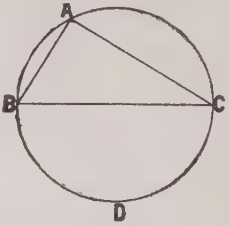
But by Case I, angle B A D is measured by one-half arc B D, and angle C A D is measured by one-half arc C D.

Hence angle B A C is measured by one-half (arc B D — arc C D), or by one-half arc B C.



164. *Corollary I. All the angles A, B, C, inscribed in the same segment are equal.*

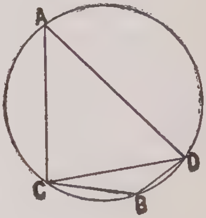
For each is measured by one-half arc D E.



165. *Corollary II. An angle B A C inscribed in a semi-circle is a right angle.*

For it is measured by one-half the semi-circumference B D C; that is, by a quadrant (Article 145.)

166. *Corollary III. An angle as C A D, inscribed in a segment greater than a semi-circle is acute.*



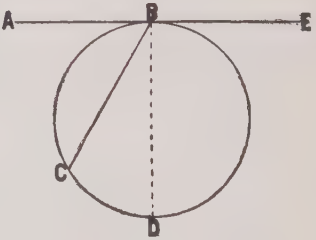
For it is measured by one-half an arc less than a semi-circumference, that is, by an arc less than a quadrant

An angle C B D inscribed in a segment less than a semi-circle is obtuse.

For it is measured by one-half an arc greater than a semi-circumference; that is bv an arc greater than a quadrant.

THEOREM XLVI.

167. *An angle formed by a tangent and a chord is measured by one-half the intercepted arc.*



Let A E be tangent to the circumference B C D at B, and let B C be a chord.

To prove that angle A B C is measured by one-half arc B C.

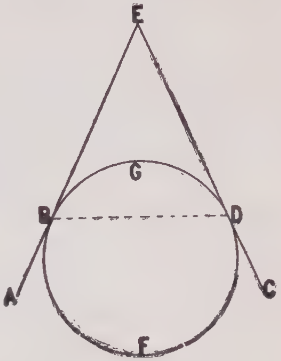
Draw the diameter B D perpendicular to A E (Theorem XLIII, Corollary I.) Then angle A B C = angle A B D — angle C B D. But since a right angle is measured by one-half a semi-circumference, angle A B D is measured by one-half arc B C D. And (by Theorem XLV), angle C B D is measured by one-half arc C D.

Therefore angle A B C is measured by one-half (arc B C D— arc C D), or by one-half arc B C.

Similarly, angle C B E is measured by one-half arc C D B.

THEOREM XLVII.

168. *The angle between two tangents is measured by one-half the difference of the intercepted arcs.*



Let A E and C E be tangent to the circumference B D F at B and D respectively.

To prove that angle E is measured by one-half (arc B F D — arc B G D.)

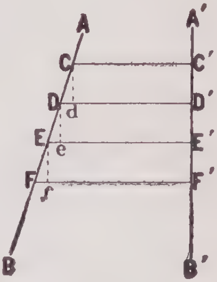
Draw the chord B D. Then the exterior angle A B D is equal to the sum of the opposite interior angles E and B D E. That is, angle E = angle A B D — angle B D E.

But (by Theorem XLVI), angle A B D is measured by one-half arc B F D, and angle B D E is measured by one-half arc B G D.

Therefore angle E is measured by one-half arc B F D — arc B G D).

THEOREM XLVIII.

169. *If a series of parallels, cutting two straight lines, intercept equal distances on one of these lines, they also intercept equal distances on the other line.*



Let the lines A B and A′ B′ be cut by the series of parallels C C′, D D′, E E′, and F F′; and let C D = D E = E F.

To prove that C′ D′ = D′ E′ = E′ F′:

Draw Cd, De, and Ef, parallel to A′ B′.

Then in the triangles C Dd, D Ee, and E Ff, by hypothesis, C D = D E = E F.

Also since the parallels D D′, E E′, F F′, are cut by A B, angle C Dd = angle D Ee = angle E Ff, (Theorem V).

And since the parallels Cd, De, and Ef are cut by A B, angle D Cd = angle E De = angle F Ef (Theorem V).

Therefore the triangles are equal, (Theorem XVIII), and Cd = De = Ef.

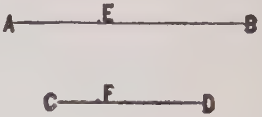
But in the parallelograms $C C' D'd$, $D D' E'e$, and $E E' F'f$, we have

$Cd = C D'$, $De = D' E'$, and $Ef = E' F'$ (Theorem XXIX).

Therefore, $C' D' = D' E' = E' F'$.

170. Definition. Two straight lines are said to be divided *proportionally* when their corresponding segments are in the same ratio as the lines themselves.

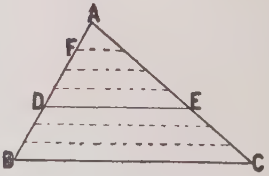
Thus the lines A B and C D are divided proportionally at E and F if $AB : CD = AE : CF$.



THEOREM XLIX.

171. *A line parallel to one side of a triangle divides the other two sides proportionally.*

Let D E be parallel to side B C of triangle A B C.



To prove that $\frac{AB}{AD} = \frac{AC}{AE}$

When A B *and* A D *are commensurable, i. e., have a common unit of measure.*

Let A F be a common measure of A B and A D, and suppose it to be contained seven times in A B and four times in A D.

$$\text{Then} \quad \frac{AB}{AD} = \frac{7}{4}. \qquad (1)$$

Drawing parallels to B C through the several points of division, the side A C will be divided into seven equal parts (Theorem XLVIII), of which A E will contain four.

$$\text{Then} \quad \frac{AC}{AE} = \frac{7}{4}. \qquad (2)$$

From (1) and (2), by axiom,

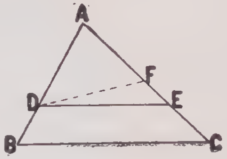
$$\frac{AB}{AD} = \frac{AC}{AE}.$$

When A B *and* A D *are incommensurable.*

This may be proved also, but depends on considerations not undertaken in this book:

THEOREM L.

172. Conversely, *a straight line which divides two sides of a triangle proportionally is parallel to the third side.*



In the triangle A B C, let D E be drawn so that $\frac{AB}{AD} = \frac{AC}{AE}$.

To prove that D E is parallel to B C:

If D E is not parallel to B C, let D F be drawn parallel to B C.

Then by (Theorem XLIX) $\frac{AB}{AD} = \frac{AC}{AF}$.

But by hypothesis, $\frac{AB}{AD} = \frac{AC}{AE}$.

Whence $\frac{AC}{AE} = \frac{AC}{AF}$.

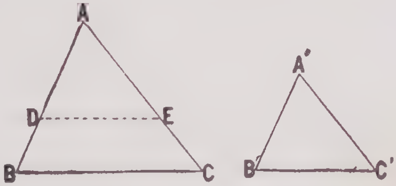
Therefore, A E = A F, which is impossible unless D F coincides with D E.

Hence, D E is parallel to B C.

173. Definition. Two polygons are called *similar* when they are mutually equiangular, and have their homologous sides proportional.

THEOREM LI.

174. *Two triangles are similar when they are mutually equiangular.*



In the triangles A B C and A′ B′ C′, let angle A = angle A′, angle B = angle B′, and angle C = angle C′.

To prove that the triangles are similar:

Superpose the triangle A′ B′ C′ upon A B C so that angle A′ shall coincide with its equal angle A, the side B′ C′ falling at D E.

Then since angle A D E = angle B, D E is parallel to B C (Theorem VI).

Hence, (by Theorem XLIX),

$$\frac{A\,B}{A\,D} = \frac{A\,C}{A\,E}, \text{ or } \frac{A\,B}{A'\,B'} = \frac{A\,C}{A'\,C'}. \qquad (1)$$

Similarly, by superposing the triangle A′ B′ C′ upon A B C so that angle B′ shall coincide with angle B, it may be proved that

$$\frac{A\,B}{A'\,B'} = \frac{B\,C}{B'\,C'}. \qquad (2)$$

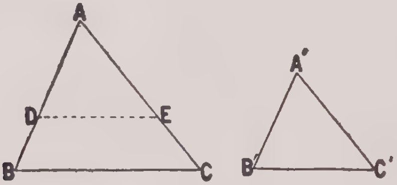
From (1) and (2) we have

$$\frac{A\,B}{A'\,B'} = \frac{A\,C}{A'\,C'} = \frac{B\,C}{B'\,C'}.$$

Therefore, the homologous sides are proportional and the triangles are similar.

THEOREM LII.

175. *Two triangles are similar when their homologous sides are proportional.*



In the triangles A B C and A′ B′ C′, let

$$\frac{A\,B}{A'\,B'} = \frac{A\,C}{A'\,C'} = \frac{B\,C}{B'\,C'}$$

To prove that the triangles are similar.

Take A D = A′ B′, and A E = A′ C′, and join D E. Then from the given proportion we have

$$\frac{A\,B}{A\,D} = \frac{A\,C}{A\,E}$$

Therefore the line D E is parallel to B C (Theorem L.)

Then since the parallels D E and B C are cut by A B, angle A D E = angle B, and angle A E D = angle C.

Hence the triangles A B C and A D E are mutually equiangular, and therefore similar (Theorem LI.)

Then their homologous sides are proportional; that is,

$$\frac{A\,B}{A\,D} = \frac{B\,C}{D\,E}, \text{ or } \frac{A\,B}{A'\,B'} = \frac{B\,C}{D\,E}$$

But by hypothesis, $\frac{A\,B}{A'\,B'} = \frac{B\,C}{B'\,C'}$

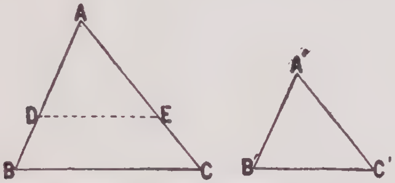
Hence D E = B′ C′, and the triangles A D E and A′ B′ C′ are mutually equilateral, and therefore equal (Theorem XIX.)

But the triangle A D E has been proved similar to A B C.

Hence the triangle A′ B′ C′ is similar to A B C.

THEOREM LIII.

176. *Two triangles are similar when they have an angle of one equal to an angle of the other, and the sides including these angles proportional.*



In the triangles A B C and A′ B′ C′, let angle A = angle A and $\frac{A\,B}{A'\,B'} = \frac{A\,C}{A'\,C'}$

To prove that the triangles are similar.

Superpose the triangle A′ B′ C′ upon A B C in such a way that the angle A′ shall coincide with its equal A, the side B′ C′ falling at D E.

Then by hypothesis, $\frac{A\,B}{A\,D} = \frac{A\,C}{A\,E}$

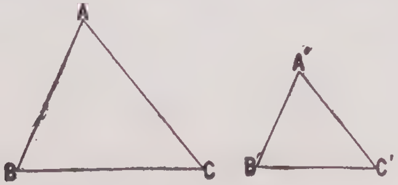
Therefore the side D E is parallel to B C (Theorem L), and angle A D E = angle B and angle A E D = angle C.

Hence the triangles A B C and A′ B′ C′ are mutually equiangular, and therefore similar.

That is, the triangles A B C and A′ B′ C′ are similar.

THEOREM LIV.

177. *Two triangles are similar when the sides of one are parallel or perpendicular respectively to the sides of the other.*



In the triangles A B C and A′ B′ C′, let A′ B′ be parallel or perpendicular to A B, A′ C′ to A C, and B′ C′ to B C.

To prove that the triangles are similar.

Since the sides of the angles A and A′ are parallel or perpendicular each to each, the angles are either equal or supplementary, for they have the same differences of direction (Theorem IV); and the same is true of the angles B and B′, C and C′.

We may then make three hypotheses with regard to the angles of the triangles, R denoting a right angle :—

FIRST. If supplementary.	SECOND. If one pair is equal and two supplementary.	THIRD. If equal.
$A + A' = 2\,R$	$A = A'$	$A = A'$
$B + B' = 2\,R$	$B + B' = 2\,R$	$B = B'$
$C + C' = 2\,R$	$C + C' = 2\,R$	$\therefore C = C'$

(Theorem XVI, Corollary II.)

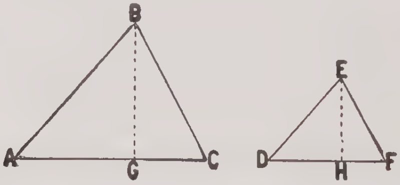
But the first and second hypotheses are impossible, since, in either case, the sum of the angles of the two triangles would exceed four right angles.

We can then only have $A = A'$, $B = B'$, and $C = C'$.

Hence the triangles are mutually equiangular, and therefore similar. (Theorem LI.)

THEOREM LV.

178. *The homologous altitudes of two similar triangles are proportional to the homologous sides.*



In the similar triangles A B C and D E F, let B G and E H be homologous altitudes

To prove that

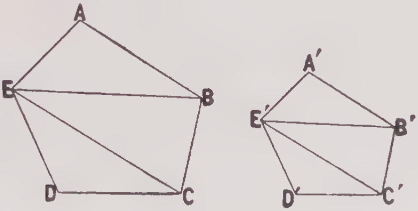
$$\frac{BG}{EH} = \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

The two right triangles A B G and D E H are mutually equiangular (Theorem XVI, Corollary II), and hence similar (Theorem LI.)

Therefore, $\frac{BG}{EH} = \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

THEOREM LVI.

179. *Two polygons are similar when they are composed of the same number of triangles, similar each to each and similarly placed.*



In the polygons A — E and A′ — E′, the triangle A B E will be similar to A′ B′ E′, B C E to B′ C′ E′, and C D E to C′ D′ E′.

To prove that the polygons are similar.

I. The polygons are mutually equiangular.

For since two similar triangles are mutually equiangular,

angle A = angle A′.

Also angle A B E + angle E B C = angle A′ B′ E′ + angle E′ B′ C′.

Or, angle A B C = A′ B′ C′.

In like manner, angle B C D = angle B′ C′ D′.

And angle D = angle D′.

II. The polygons have their homologous sides proportional.

For since two similar triangles have their homologous sides proportional,

$$\frac{AB}{A'B'} = \frac{BE}{B'E'}, \text{ and } \frac{BE}{B'E'} = \frac{BC}{B'C'}$$

Whence, $\frac{AB}{A'B'} = \frac{BC}{B'C'}$

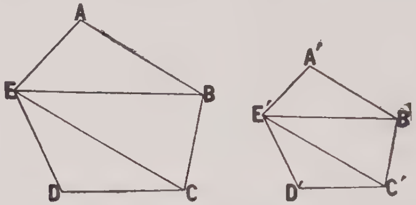
In like manner we may prove,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

Therefore the polygons are similar, (article 173.)

THEOREM LVII.

180. Conversely, *two similar polygons may be decomposed into the same number of triangles, similar each to each and similarly placed.*



Let E and E′ be homologous vertices of the similar polygons A — E and A′ — E′, and draw the diagonals E B, E C, E′ B′, and E′ C′.

To prove that the triangle A B E is similar to A′ B′ E′, B C E to B′ C′ E′, and C D E to C′ D′ E′.

Since the polygons are similar (Article 173), angle A = angle A′, and $\frac{AE}{A'E'} = \frac{AB}{A'B'}$

Therefore the triangles A B E and A′ B′ E′ are similar, for they have an angle of one equal to an angle of the other, and the sides including these angles proportional (Theorem LIII).

Again, since the polygons are similar, angle A B C = angle A′ B′ C′.

And since the triangles A B E and A′ B′ E′ are similar, angle A B E = angle A′ B′ E′.

Subtracting these equals, we have angle A B C — angle A B E = angle A′ B′ C′ — angle A′ B′ E′, or angle E B C = angle E′ B′ C′.

Also, from the similarity of the polygons,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}$$

And since the triangles A B E and A′ B′ E′ are similar,

$$\frac{AB}{A'B'} = \frac{BE}{B'E'}$$

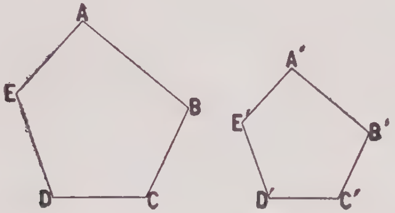
Whence, $\frac{BC}{B'C'} = \frac{BE}{B'E'}$

Therefore the triangles B C E and B′ C′ E′ are similar (Theorem LIII).

In like manner it may be proved that the triangles C D E and C′ D′ E′ are similar.

THEOREM LVIII.

181. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*



Let P and P′ denote the perimeters of the similar polygons A — E, and A′ — E′.

To prove that $\frac{P}{P'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc.

Since the polygons are similar, we have,

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

Whence by Article 139,

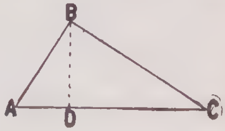
$$\frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

That is $\frac{P}{P'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$, etc.

THEOREM LIX.

182. *In a right triangle the perpendicular drawn from the vertex of the right angle to the hypothenuse divides the triangle into two triangles similar to the whole triangle and to each other.*

In the right triangle ABC, let B D be drawn from the vertex B of the right angle perpendicular to the hypothenuse A C.



To prove that the two triangles A B D and B D C are similar to the triangle A B C and to each other.

The two right triangles A B D and A B C have the acute angle A common; they are therefore mutually equiangular (Theorem XVI, Corollary II), and similar (Theorem LI).

The two right triangles A B C and B C D have the acute angle C common; therefore, they are mutually equiangular and similar.

The two triangles A B D and B C D, being similar each to A B C, are similar to each other.

183. *Corollary I.* Since A B C and A B D are similar triangles,

$$\frac{AC}{AB} = \frac{AB}{AD}$$

And since A B C and B C D are similar,

$$\frac{AC}{CB} = \frac{CB}{CD},$$

that is, *if in a right triangle, a perpendicular is drawn from the vertex of the right angle to the hypothenuse, either side about the right angle is a mean proportional between the whole hypothenuse, and the adjacent segment.*

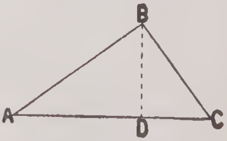
184. *Corollary II.* As A B D and B C D are similar triangles,

$$\frac{AD}{DB} = \frac{DB}{DC},$$

that is, *in a right triangle the perpendicular from the vertex of the right angle is a mean proportional between the segments of the hypothenuse.*

THEOREM LX.

185. *The square described on the hypothenuse of a right triangle is equivalent to the sum of the squares described upon the other sides.*



Let A B C be a right triangle.

To prove that $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$.

Draw B D perpendicular to A C.

Now (by Theorem LIX, Corollary I),

$$AC : AB = AB : AD.$$

That is, $\overline{AB}^2 = AC \times AD$.

And in the same way,

$$\overline{BC}^2 = AC \times CD.$$

Adding these equations,

$$\begin{aligned}\overline{A\,B}^2 + \overline{B\,C}^2 &= A\,C \times A\,D + A\,C \times C\,D.\\ &= A\,C \times (A\,D + C\,D)\\ &= A\,C \times A\,C\\ &= \overline{A\,C}^2\end{aligned}$$

Therefore, $\overline{A\,B}^2 + \overline{B\,C}^2 = \overline{A\,C}^2$

186. Corollary I. Since

$$\overline{A\,C}^2 = \overline{A\,B}^2 + \overline{B\,C}^2,$$

$$\overline{B\,C}^2 = \overline{A\,C}^2 - \overline{A\,B}^2$$

And $B\,C = \sqrt{A\,C^2 - A\,B^2}$.

AREAS OF POLYGONS.

DEFINITIONS.

187. The **Area** of a polygon is the measure of its surface. It is expressed in units, which represent the number of times the polygon contains the square unit that is taken as a standard. This unit is called the *unit of surface*, and is usually adopted arbitrarily.

188. The usual unit of surface is the square whose side is some *unit of length* ; for example, a *square inch*, or a *square foot*.

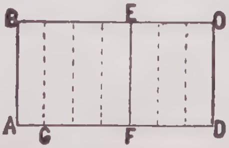
189. **Equivalent Polygons** are those which have the same area.

190. The *projection of a point* upon a straight line of indefinite length is the foot of the perpendicular drawn from the point to the line. An entire line may be said to have a *projection* upon another line in the same way.

191. A *medial line* of a triangle is a straight line drawn from either vertex to the middle point of the opposite side.

THEOREM LXI.

192. *Two rectangles having equal altitudes are to each other as their basis.*



Let A B C D and A B E F be two rectangles having equal altitudes.

To prove that $\frac{ABCD}{ABEF} = \frac{AD}{AF}$

Let A G be a common measure of A D and A F, and suppose it to be contained seven times in A D and four times in A F.

Then $$\frac{AD}{AF} = \frac{7}{4} \qquad (1)$$

Drawing perpendiculars to A D through the several points of division, the rectangle A B C D will evidently be divided into seven equal parts, of which the rectangle A B E F will contain four.

Then $$\frac{ABCD}{ABEF} = \frac{7}{4} \qquad (2)$$

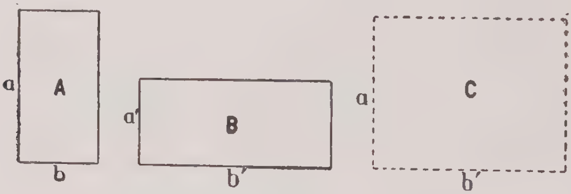
Then by (1) and (2) $\frac{ABCD}{ABEF} = \frac{AD}{AF}$, (things equal to the same thing are equal to each other, axiom.)

This may also be proved when A D and A F are incommensurable.

Corollary. Since either side of a rectangle may be taken as a base, it follows that *two rectangles with equal bases are to each other as their altitudes.*

THEOREM LXII.

193. *Any two rectangles are to each other as the products of their bases by their altitudes.*



Let A and B be any two rectangles having the altitudes a and a', and the bases b and b', respectively.

To prove that $\frac{A}{B} = \frac{a \times b}{a' \times b'}$

Let C be a third rectangle, having its altitude equal to the altitude of A and its base equal to the base of B.

Then the rectangles A and C, having equal altitudes, are to each other as their bases, (by Theorem LXI); that is,

$$\frac{A}{C} = \frac{b}{b'} \qquad (1)$$

And the rectangles C and B, having equal bases, are to each other as their altitudes (Theorem LXI, Corollary); that is,

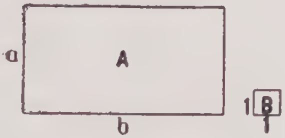
$$\frac{C}{B} = \frac{a}{a'} \qquad (2)$$

Multiplying equations (1) and (2), we have

$$\frac{A}{B} = \frac{a \times b}{a' \times b'}$$

THEOREM LXIII.

194. *The area of a rectangle is equal to the product of its base and altitude.*



Let a and b be the numerical measures of the altitude and base of the rectangle A, and let B be the square whose side is the unit of length.

To prove that the area of A, referred to B as a unit, is equal to a times b.

Since any two rectangles are to each other as the products of their bases by their altitudes (Theorem LXII), we have

$$\frac{A}{B} = \frac{a \times b}{1 \times 1} = a \times b$$

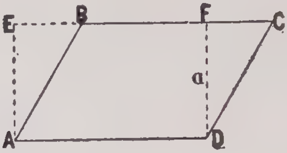
But since B is the unit of surface, $\frac{A}{B}$ is the area of A. (Article 186).

Hence, area of A $= a \times b$.

195. *Corollary.* The area of a square is equal to the square of one of its sides.

THEOREM LXIV.

196. *The area of a parallelogram is equal to the product of its base and altitude.*



Let A B C D be a parallelogram leaving its altitude D F equal to a and its base A D equal to b.

To prove that A B C D $= a \times b$:

Draw A E perpendicular to A D and meeting B C produced at E.

Then A E F D is a rectangle having its altitude equal to a and its base equal to b.

In the right triangles A B E and D C F, A B = D C and A E = D F (Theorem XXIX).

Therefore, the triangles are equal (Theorem XX). Now if from the entire figure A E C D, we take the triangle A B E, there remains the parallelogram A B C D; and if we take the triangle D C F, there remains the rectangle A E F D.

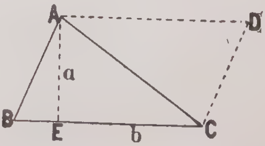
Therefore, area A B C D = area A E F D.

But the area of a rectangle A E F D $= a \times b$ (Theorem LXIII).

Hence, area A B C D $= a \times b$.

THEOREM LXV.

197. *The area of a triangle is equal to one-half the product of its base and altitude.*



Let A B C be a triangle, having its altitude equal to a and its base equal to b.

To prove that area A B C $= \frac{1}{2} a \times b$:

Draw the lines A D and C D parallel to B C and A B.

Then A B C D is a parallelogram having its altitude equal to a and its base equal to b.

Now a diagonal of a parallelogram divides it into equal triangles (Theorem XXIX).

Therefore area A B C $= \frac{1}{2}$ area A B C D.

But the area of the parallelogram A B C D is equal to a times b, (Theorem LXIV).

Hence, area A B C $= \frac{1}{2} a \times b$.

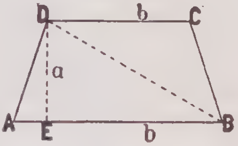
198. *Corollary I.* Two triangles having equal bases and equal altitudes are equivalent.

199. *Corollary II.* Two triangles having equal altitudes are to each other as their bases; two triangles having equal bases are to each other as their altitudes; and any two triangles are to each other as the products of their bases by their altitudes.

200. *Corollary III.* A triangle is equivalent to one-half a parallelogram having the same base and altitude.

THEOREM LXVI.

201. *The area of a trapezoid is equal to one-half the sum of its parallel sides multiplied by its altitude.*



Let A B C D be a trapezoid, having its altitude equal to a, and its parallel sides equal to b and b' respectively.

To prove that area A B C D $= a \times \frac{1}{2} (b + b')$:

Draw the diagonal B D dividing the trapezoid into two triangles A B D and B C D, having the common altitude a and their bases equal to b and b' respectively.

Then, area A B C D = area A B D + area B C D.

But since the area of a triangle is equal to one-half the product of its base by its altitude, (Theorem LXV),

area A B D $= \frac{1}{2} a \times b$,

and area B C D $= \frac{1}{2} a \times b'$.

Therefore, area A B C D $= \frac{1}{2} a \times b + \frac{1}{2} a \times b'$,

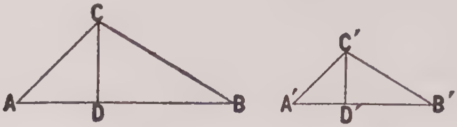
$= a \times \frac{1}{2} (b + b')$.

202. *Corollary.* Since the line joining the middle points of the non-parallel sides of a trapezoid is equal to one-half the sum of the bases (Theorem XXXV), it follows that,

The area of a trapezoid is equal to the product of its altitude by the line joining the middle points of the non-parallel sides.

THEOREM LXVII.

203. *Two similar triangles are to each other as the squares of their homologous sides.*



Let A B and A′ B′ be homologous sides of the similar triangles A B C and A′ B′ C′.

To prove that $\frac{ABC}{A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$

Draw the altitudes C D and C′ D′.

Then since any two triangles are to each other as the products of their bases by their altitudes (Theorem LXV, Corollary II), we have

$$\frac{ABC}{A'B'C'} = \frac{AB \times CD}{A'B' \times C'D'} = \frac{AB}{A'B'} \times \frac{CD}{C'D'} \quad (1)$$

But the homologous altitudes of two similar triangles are in in the same ratio as any two homologous sides (Theorem LV); hence,

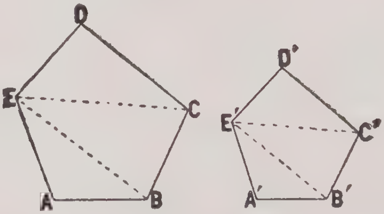
$$\frac{CD}{C'D'} = \frac{AB}{A'B'}$$

Substituting in (1), we have

$$\frac{ABC}{A'B'C'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$$

THEOREM LXVIII.

204. *Two similar polygons are to each other as the squares of their homologous sides.*



Let K and K′ denote the areas of the similar polygons **A — E** and A′ — E′.

To prove that $\frac{K}{K'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}$

Draw the diagonals E B, E C, E′ B′, and E′ C′.

Then since the triangle A B E is similar to A′ B′ E′, (Theorem LVII), we have

$$\frac{A\ B\ E}{A'\ B'\ E'} = \frac{\overline{A\ B}^2}{\overline{A'\ B'}^2} \text{ (Theorem LXVII).}$$

In like manner,

$$\frac{B\ C\ E}{B'\ C'\ E'} = \frac{\overline{B\ C}^2}{\overline{B'\ C'}^2} \text{ and } \frac{C\ D\ E}{C'\ D'\ E'} = \frac{\overline{C\ D}^2}{\overline{C'\ D'}^2}.$$

But from the similarity of the given polygons,

$$\frac{\overline{A\ B}^2}{\overline{A'\ B'}^2} = \frac{\overline{B\ C}^2}{\overline{B'\ C'}^2} = \frac{\overline{C\ D}^2}{\overline{C'\ D'}^2}$$

Therefore,

$$\frac{A\ B\ E}{A'\ B'\ E'} = \frac{B\ C\ E}{B'\ C'\ E'} = \frac{C\ D\ E}{C'\ D'\ E'}$$

Whence,

$$\frac{A\ B\ E + B\ C\ E + C\ D\ E}{A'\ B'\ E' + B'\ C'\ E' + C'\ D'\ E'} = \frac{A\ B\ E}{A'\ B'\ E'} = \frac{\overline{A\ B}^2}{\overline{A'\ B'}^2}$$

That is,

$$\frac{K}{K'} = \frac{\overline{A\ B}^2}{\overline{A'\ B'}^2}$$

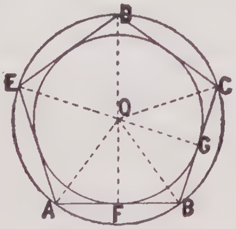
by ratio and proportion (Theorem IX).

REGULAR POLYGONS AND MEASUREMENT OF THE CIRCLE.

205. Definition. *A regular polygon is one which is both equilateral and equiangular.*

THEOREM LXIX.

206. *A circle may be circumscribed about, or inscribed within, any regular polygon.*



Let A B C D E be a regular polygon.

I. To prove that a circle may be circumscribed about it:

Let O be the centre of the circle passing through the vertices A, B, and C (the point O, at which perpendiculars erected at the middle points of A B and B C intersect, will be the centre of such a circle, A O, B O, and C O, being equal, and therefore radii (by Theorem XI); join O A, O B, O C, and O D.

Then since the polygon is equiangular, angle A B C = angle B C D.

And since the triangle O B C is isosceles, angle O B C = angle O C B.

Subtracting these equals, we have angle O B A = angle O C D.

Also, O B = O C, being equal radii; and since the polygon is equilateral, A B = C D.

Therefore the triangles O A B and O C D are equal (Theorem XVIII) and O A = O D.

Hence the circle passing through A, B, and C also passes through D.

In like manner it may be shown that the circle passing through B, C, and D, also passes through E.

Therefore a circle may be circumscribed about A B C D E.

II. To prove that a circle may be inscribed in A B C D E.

Since the sides A B, B C, C D, etc., are equal chords of the circumscribed circle, they are equally distant from the centre.

For, drawing O F and O G perpendicular to A B and B C respectively, we have F as the middle point of A B, and G of B C, (by Theorem XLII), and the right triangle O F B and O G B are equal (by Theorem XX.)

Hence O F = O G and the circle described with O as a centre and O F as a radius will be inscribed in the polygon A D C D E.

207. Definitions. The *radius* of a regular polygon is the radius of its circumscribed circle, as O A.

The *apothem* is the radius of the inscribed circle, as O F.

The *centre* is the common centre O of the circumscribed and inscribed circles.

The *angle at the centre* is the angle, as A O B, between radii drawn to the extremities of any side.

208. *Corollary. Each angle at the centre of a regular polygon is equal to four right angles divided by the number of sides.*

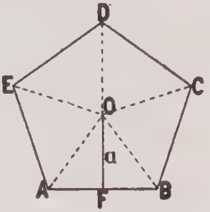
For from the equal triangles O A B, O B C, etc., we have, angle A O B = angle B O C = angle C O D, etc.

Therefore each angle is equal to four right angles divided by the number of sides (Theorem I.)

THEOREM LXX.

209. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*

Let a denote the apothem O F, and P the perimeter of the regular polygon A B C D E.



To prove that area A B C D E = one-half P × a.

By drawing the radii O A, O B, O C, etc., the polygon may be divided into a series of triangles O A B, O B C, etc., whose common altitude is a.

Then (by Theorem LXV),

Area O A B = one-half A B × a.
Area O B C = one-half B C × a, and so on.

Adding, we have,

Area O A B + area O B C + etc., = one-half (A B + B C + etc.) × a.

That is, area A B C D E = one-half P × a.

The following important theorem is given more in the light of an example than a proof.

THEOREM LXII.

I. An inscribed or circumscribed polygon of an infinite number of sides practically coincides with the inscribing or circumscribing circle.*

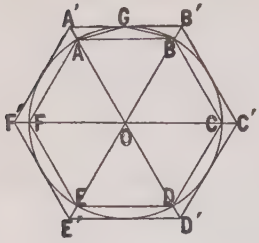
II. The areas of the above polygons are practically the same as that of the circle.

III. The approximate ratio of the circumference of a circle to its diameter is 3.1416.

IV. The area of the circle is 3.1416 *times the square of the radius.*

---

* To speak mathematically the perimeter of the inscribed or circumscribed polygon approaches the circumference of the circle and the areas of the polygons approach the area of the circle as a limit as the number of sides of each polygon is indefinitely increased. The demonstration here given will, however, suffice for the purposes of this book.



I. Let the number of sides of the inscribed and circumscribed polygons be doubled. It is readily seen that the sides of the polygons approach more closely the circumference, as is seen by replacing the side A B by the sides A G and B G of a polygon containing twice as many sides as the polygon A — F. If the number of sides were infinite, there would be no appreciable space between the circumference and the perimeter of the polygon. Hence they would practically coincide.

II. The area of the polygon A F is evidently equal to the sum of the triangles A O B, B O C, etc. Now if the number of sides of the polygon is increased indefinitely it is evident, as there is no appreciable area between the perimeter and circumference (I), that the sum of the areas of the infinite number of triangles thus formed is practically equal to the area of the circle.

III. By means of formulas, which we will not here attempt to derive, it is found that the perimeter of a polygon of 512 sides circumscribed about a circle whose diameter is 1, is 3.14163 while the perimeter of a 512 sided polygon inscribed within the above circle is 3.14157. The greater the number of sides to the polygons the nearer do these values approach each other. The value of the circumference must be between these values. The above figures are sufficient for our purpose. We, therefore, take as our approximate ratio of the circumference to the diameter 3.1416, i.e., denoting the circumference by C and the diameter by D we have $\frac{C}{D} = 3.1416$ or $C = 3.1416\ D$. The ratio is usually designated by the Greek letter Pi, π. Hence we say $C = \pi\ D$ or the circumference is π times the diameter. This holds for all circles.

IV. The area of a polygon is equal to one-half the value of the product of the perimeter and apothem. In an inscribed or circumscribed polygon, the apothem becomes practically equal to the radius of the circle when the number of sides of the polygon becomes infinite.

Hence the area of any circle is equal to one-half the circumference (π D) times the radius (which we will denote by r). As the diameter is twice the radius, we have $D = 2\ r$. Hence the area of the circle is $\frac{1}{2} \times \pi\ 2\ r \times r = \pi\ r^2$. Writing $\frac{D}{2}$ in place of r we have $\frac{1}{4}\ \pi\ D^2$, the approximate value of π being 3.1416

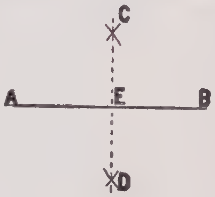
PROBLEMS OF CONSTRUCTION.

We have hitherto assumed that our figures were already constructed. The Problems of Construction here following depend for their solution upon principles already explained; and therefore, when the demonstration is not given in full, the theorems necessary to prove the construction will be referred to in the order in which they are to be used, and the student must complete the demonstration.

PROBLEM I.

1. *To bisect a given straight line.*

Let A B be the given straight line. From A and B as centres, with a radius greater than half of A B, describe arcs cutting one another at C and D; join C and D cutting A B at E, and the line A B is bisected at E.

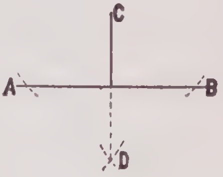


For C and D being each equally distant from A and B, the line C D must be perpendicular to A B at its middle point. (Article 45.)

PROBLEM II.

2. *From a given point without a straight line to draw a perpendicular to that line.*

Let C be the point and A B the line. From C as a centre, describe an arc cutting A B in two points A and B; with A and B as centres, with a radius greater than half A B, describe arcs intersecting at D. Draw C D, and it is the perpendicular required. (Converse of Theorem VII.)

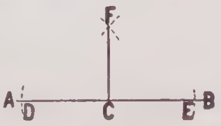


PROBLEM III.

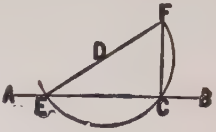
3. *From a given point in a straight line to erect a perpendicular to that line.*

Let C be the given point and A B the given line.

With C as a centre describe an arc cutting A B in D and E; with D and E as centres, with a radius greater than D C, describe arcs intersecting at F. Draw C F, and it is the perpendicular required. (Converse of Theorem VII.)



Second Method. With any point, D, without the line A B, with a radius equal to the distance from D to C, describe an arc cutting A B at E; draw the diameter E D F. Draw C F, and it is the perpendicular required. (Theorem XLV.)

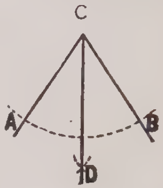


PROBLEM IV.

4. *To bisect a given arc, or angle.*

1st. Let A B be the given arc. Draw the chord A B and bisect it with a perpendicular.

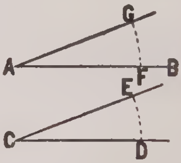
2nd. Let C be the given angle. With C as a centre, describe an arc cutting the sides of the angle in A and B; bisect the arc with the line C D, and it will also bisect the angle C. (Theorem XXXVII.)



PROBLEM V.

5. *At a given point in a straight line to make an angle equal to a given angle.*

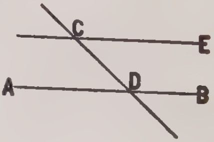
Let A be the given point in the line A B, and C the given angle. With C as a centre describe an arc D E cutting the sides of the angle C; with A as a centre, with the same radius, describe an arc F G; with F as a centre, with a radius equal to the distance from D to E, describe an arc cutting the arc F G. Draw A G. The angle A = angle C. (Theorem XXXVII and XXXIX.)



PROBLEM VI.

6. *Through a given point to draw a line parallel to a given straight line.*

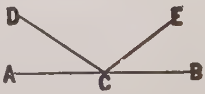
Let C be the given point, and A B the given line. From C draw a line C D to A B; at C in the line C D make an angle D C E equal to angle C D A (by Problem V;) C E is parallel to A B. (Theorem VI.)



PROBLEM VII.

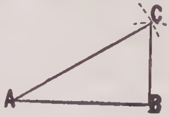
7. *Two angles of a triangle given to find the third.*

Draw an indefinite line A B; at any point C make an angle A C D equal to one of the given angles, and angle D C E equal to the other (by Problem V.) Then E C B is equal to the third angle. (Theorems I and XVI.)



PROBLEM VIII.

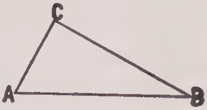
8. *The three sides of triangle given, to construct the triangle.*



Take A B equal to one of the given sides. With A as a centre, with a radius equal to another of the given sides, describe an arc; and with B as a centre, with a radius equal to the third side, describe an arc intersecting the first arc at C. Draw A C and B C, and A C B is evidently the triangle required.

PROBLEM IX.

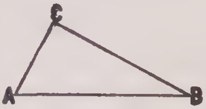
9. *Two sides and the included angle of a triangle given, to construct the triangle.*



Draw A B equal to one of the given sides, at B make the angle A B C equal to the given angle (by Problem V), and take B C equal to the other given side; join A C, and A B C is evidently the triangle required.

PROBLEM X.

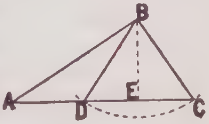
10. *Two angles and a side of a triangle given, to construct the triangle.*



If the angles are not both adjacent to the given sides, find the third angle (by Problem VII.) Then draw A B equal to the given side, and at B make an angle A B C equal to one of the angles adjacent to A B, and at A make an angle B A C equal to the other angle adjacent to A B, and A B C is evidently the triangle required.

PROBLEM XI.

11. *Two sides of a triangle and the angle opposite one of them given, to construct the triangle.*



Draw an indefinite line A C, at A make an angle C A B equal to the given angle, and take A B equal to side adjacent to the given angle; with B as a centre, with a radius equal to the other given side, describe an arc cutting A C.

If the angle A is acute.

1st. The given side B C, opposite the given angle, may be less than the other given side, then the arc described from B as a centre will cut A C in two points, C and D, on the same side of A, and, drawing B C and B D, the triangles A B C and A B D (whose angle B D A is the supplement of the angle B C A), both satisfy the given conditions.

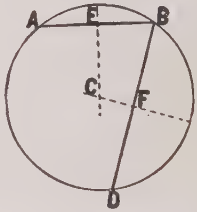
2nd. The given side opposite the given angle may be equal to the perpendicular B E; then the arc described from B as a centre will touch A C, and the right triangle A B E is the only one that can satisfy the given conditions.

3rd. The side opposite the given angle may be greater than the other given side; then the arc described from B as a centre will cut A C in C, and in another point on the other side of A. In this case there can be but one triangle A B C satisfying given conditions, the triangle formed on the opposite side of A B containing not the given angle, but its supplement.

4th. If the given angle is not acute, the given side opposite the given angle must be greater than the other given side, and, as in the last case above, there can be but one solution.

PROBLEM XII.

12. *To find the centre of a given circumference or of a given arc.*



Let A B D be the given circumference, or arc.

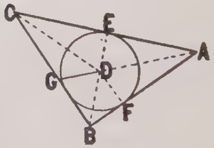
Draw any two chords not parallel to each other, as A B, B D, and bisect these chords by the perpendiculars C E and C F.

These perpendiculars will intersect at the center of the circumference, or arc. (Theorem XLII).

Scholium. By the same construction a circumference may be made to pass through any three given points; or a circle circumscribed about a given triangle.

PROBLEM XIII.

13. *To inscribe a circle in a given triangle.*



Let A B C be the given triangle.

Bisect any two of its angles. With the point D, where the two bisecting lines meet, as a centre, with a radius equal to the distance of D from any one of the sides, describe a circle, and it will be the one required.

Draw the perpendiculars D E, D F, and D G. The angles at A are equal by construction, and the angles A E D and A F D are each right angles; therefore the triangles A D E and A F D are mutually equiangular (Theorem XVI), and the hypothenuse A D is common; therefore the triangles are equal (Theorem XVII), and D E = D F. In like manner D E = D G.

Therefore, the circle described from D as a centre with the radius D E will pass through the points F and G; and since the angles at E, F, and G, are right angles, the sides of the triangle A B C are tangents; therefore the circle E F G is inscribed in the triangle A B C.

Scholium. The lines bisecting the angles of a triangle all meet in the same point.

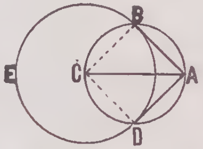
PROBLEM XIV.

14. *Through a given point to draw a tangent to a given circumference.*

1st. If the given point is in the circumference. Erect a perpendicular to the radius at the given point.

2nd. If the given point is without the circumference.

Join the given point A with the centre C of the given circle B D E; on A C as a diameter describe a circle cutting the given circle in B and D. Draw A B and A D, and each will be tangent to the given circle through the given point.



For drawing the radii C B and C D, the angles B and D are each right angles (Theorem XLV), and therefore A B and A D are tangents to the given circle.

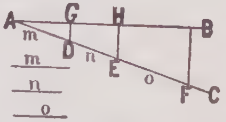
Corollary. The tangents A B and A D are equal.

PROBLEM XV.

15. *To divide a given line into parts proportional to given lines.*

Let it be required to divide A B into parts proportional to m, n, and o.

Draw at any angle with A B an indefinite line A C. From A cut off A D, D E, and E F respectively equal to m, n, and o. Join B to F and through D and E draw lines parallel to B F. These parallels divide the line as required. (Theorem XLIX).



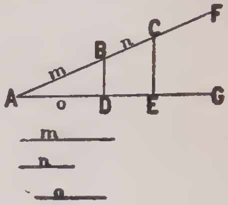
PROBLEM XVI.

16. *To find a fourth proportional to three given lines.*

Let it be required to find a fourth proportional to M, N, and O.

Draw at any angle with the other the indefinite lines A F, and A G.

From A F cut off A B = m, B C = n, and from A G cut off A D = o. Join B D and through C draw C E parallel to B D; then D E is the required fourth proportional (Theorem XLIX).

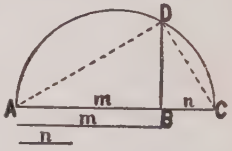


PROBLEM XVII.

17. *To find a mean proportional between two given lines.*

Let it be required to find a mean proportional between m and n.

From an indefinite line cut off A B = *m*, B C = *n*; on A C as a diameter describe a semi-circle, and at B draw B D perpendicular to A C. B D is the mean proportional required. Join A D, and D C. (Section 184).



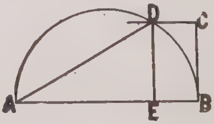
18. Definition. When a line is divided so that one segment is a mean proportional between the whole line and the other segment, it is said to be divided *in extreme and mean ratio.*

PROBLEM XVIII.

19. *To construct a parallelogram, having the sum of its base and altitude given, which shall be equivalent to a given square.*

On A B, the given sum, as a diameter, describe a semi-circumference.

At any point, as B, in A B, draw the perpendicular B C equal to a side of the given square; through C draw C D parallel to A B, cutting the circumference in D; draw D E perpendicular to A B. A E and E B are respectively the base and the altitude of the parallelogram required (Problem XVII).

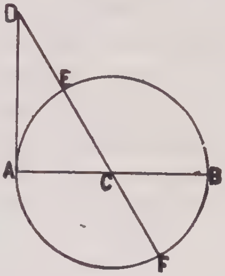


Scholium. If the side of the square is greater than half the sum of the base and the altitude, the construction is impossible.

PROBLEM XIX.

20. *To construct a parallelogram having the difference between its base and altitude given, which shall be equivalent to a given square.*

On A B the given difference, as a diameter, describe a semi-circumference. At A draw the perpendicular A D equal to a side of the given square; join D with the centre C, and produce D C to F. D F and D E are respectively the base and the altitude of the parallelogram required.



PROBLEM XX.

21. *To construct a square equivalent to a given parallelogram.*

Find a mean proportional between the altitude and base of the given parallelogram (Problem XVII), and it will be a side of the required square.

PROBLEM XXI.

22. *To construct a square equivalent to a given triangle.*

Find a mean proportional between the base and half the altitude (Problem XVII), and it will be a side of the required square.

PROBLEM XXII.

23. *To construct a square equivalent to a given circle.*

Find a mean proportional between the radius and the semi-circumference, and it will be a side of the required square.

PROBLEM XXIII.

24. *To construct a square equivalent to the sum of two given squares.*

Construct a right triangle (Problem IX), with the sides adjacent to the right angle equal respectively to the sides of the given squares; the hypothenuse will be a side of the required square (Theorem LX).

Scholium. By continuing the same process we can find a square equivalent to the sum of any number of given squares.

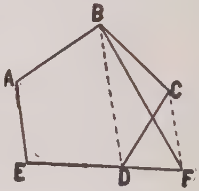
PROBLEM XXIV.

25. *To construct a square equivalent to the difference of two given squares.*

Construct a right triangle (Problem XI), taking as the hypothenuse a side of the greater square, and for one of the sides adjacent to the right angle a side of the other square; the third side of the triangle will be a side of the required square. (Theorem LX.)

PROBLEM XXV.

26. *To construct a triangle equivalent to a given polygon.*

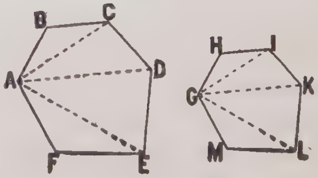


Let A — E be the polygon. Draw B D cutting off the triangle B C D; through C draw C F parallel to B D; join B F, and a polygon A B F E is formed with one side less than the given polygon and equivalent to it. For the triangles B C D and B F D, having the same base and the same altitude, are equivalent; adding to each the common part A B D E, we have A B C D E equivalent to A B F E. In like manner a polygon with one side less can be found equivalent to A B F E, and by continuing the process the sides may be reduced to three, and a triangle obtained equivalent to the given polygon,

Scholium. Since (by Problem XXI) a square can be found equivalent to a given triangle, it is evident that a square can be found equivalent to any polygon.

PROBLEM XXVI.

27. *On a given line to construct a polygon similar to a given polygon.*



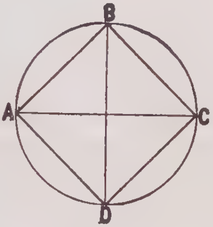
Let A D be the given polygon, and let M L be the given line.

Draw the diagonals A E, A D, and A C. At M and L make the angles G M L and G L M equal respectively to A F E and A E F, and a triangle G L M will be formed similar to A E F. In like manner on G L construct a triangle similar to A D E; on G K one similar to A C D; on G I one similar to A B C; and the polygons A D, and H G, being composed of the same number of similar triangles similarly situated, are similar. (Theorem LVI.)

PROBLEM XXVII.

28. *To inscribe a square in a given circle.*

Draw two diameters A C and B D, at right angles to each other; and join A B, B C, C D, and D A. Then A B C D is the required square. (Theorem XLV and XXXIX.)

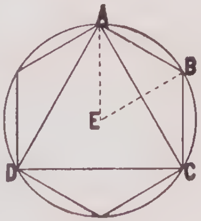


Corollary. By bisecting the arc A B, B C, C D, and D A, and drawing the chords of these smaller arcs, a regular octagon will be inscribed in the circle. By continuing this bisection regular polygons can be inscribed having the number of their sides 16, 32, 64, and so on.

PROBLEM XXVIII.

29. *To inscribe a regular hexagon in a given circle.*

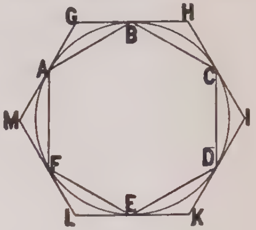
Take A B equal to the radius of the given circle, and it will be a side of the hexagon required for the angle, A B E is equilateral and therefore each of its angles has sixty degrees (Theorem XVI) and is one-sixth of four right angles. Hence A B = B C, etc, (by Theorems XXXVII and XXXIX.)



Corollary. By drawing A C, C D, and D A an equilateral triangle will be inscribed in the circle. By bisecting the arcs A B, B C, etc., and continuing this bisection, as in Problem XXVII, and drawing the chords of these smaller arcs, regular polygons can be inscribed, having the number of their sides 12, 24, 48, 96, and so on.

PROBLEM XXIX.

30. *To circumscribe about a given circle a polygon similar to a given inscribed regular polygon.*



Let A F be the given inscribed polygon. Through the points A, B, C, D, E, F draw tangents to the circumference. Those tangents intersecting will form the polygon required.

For the triangles A G B, B H C, etc., are isosceles (Problem XIV, Corollary;) and as the arcs A B, B C, etc., are equal, the angles G A B, G B A, H B C, H C B, etc., are equal (Theorem XLVI,) therefore, as the bases A B, B C, etc., are equal, those isosceles triangles are equal. Hence the angles G, H, I, K, L, M are equal, and polygon M L is equiangular; and as G B = B H = H C, etc., G H = H I, etc., therefore the polygon M I is equilateral and regular. It is also similar to A F (easily proved from Theorem LVI); and as its sides are tangents it is circumscribed about the circle.

EXAMINATION PAPER

GEOMETRY

DIRECTIONS TO THE STUDENT

In writing out this examination, use a page for each demonstration. Write the theorem at the top of the page, draw a neat figure, and describe the figure carefully. Number each statement with Roman numerals, and under each statement, in a bracket, place the reason for the statement. If your definitions do not fill a page, turn to the next page before commencing a theorem. Never give a reason simply by the number of the theorem, but be sure to give the words of the theorem or of the axiom in full. Only this will be accepted as a reason for a statement.

---

1. What is meant by *quantity?*
2. What is meant by a line?
3. Define curved line.
4. Define perpendicular.
5. Define angle.
6. Upon what does the size of an angle depend?
7. Define:
    Complementary angle.
    Supplementary angle.
8. Define a theorem.
9. Define converse of a theorem, and illustrate by an example.
10. State five axioms.

11. *Prove*—If one straight line intersects another, the vertical angles are equal.

12. If two angles of a triangle are equal, the sides opposite are equal. State and prove the converse.

13. Draw the || lines AB and CD cut by the transversal EF. Letter angles, *a*, *b*, *c*, *d*, *e*, *f*, *g*, and *h*, and state at least eight facts which are true of the equality of the different pairs of angles.

14. Prove one of the facts which you state in the above question.

15. *Prove*—If two right triangles have the hypothenuse and side of one equal to the hypothenuse and side of the other, the triangles are equal in all respects.

16. *Prove*—In an isosceles triangle the bisector of the exterior

angle at the vertex, formed by producing one of the sides, is parallel to the base.

17. Define *parallelogram.*

18. Define *rhombus.*

19. Define *trapezoid.*

20. What is the altitude of a trapezoid? Illustrate by a figure.

21. *Prove*—The diagonals of a parallelogram bisect each other.

22. Define: circle; sector; segment.

23. Define: arc; secant; tangent.

24. *Prove*—If a diameter is perpendicular to a chord, it bisects the chord.

25. *Prove*—Through any three points, provided they are not in a straight line, a circumference can be made to pass.

26. The distance from the center of a circle to a chord 16 inches long is 6 inches; find the distance from the center to a chord 10 inches long.

27. *Prove*—That an inscribed angle is measured by one-half the arc which its sides intercept.

28. *a.* The base of an isosceles triangle is 60 feet and the area is 1200 sq. ft. Find the lengths of the two equal sides.

b. What is the area of a similar triangle having a base of 30 ft.?

29. State what is meant when two lines are divided proportionally.

30. *Prove*—Similar triangles are to each other as the squares of their homologous sides.

31. Construct a rhombus, having given the base and altitude.

32. Having given two angles of a triangle, construct the third angle.

33. Define a polygon; a regular polygon.

34. In a regular polygon of 24 sides, find the value of one of the interior angles in degrees.

35. In a square field there are $5\frac{5}{8}$ acres. Compute the length of barbed wire necessary to enclose it four times.

36. B is 30 miles south of A, C is 50 miles east of A, and D is 40 miles north of C. How far is D from B?

37. The area of a pentagon is 40 sq. in. The area of a similar pentagon is 640 sq. in. If one side of the smaller polygon is 2 in., what is the length of the homologous side of the larger one?

38. The diameter of a drive-wheel is 5 ft. 3 in. How many revolutions does it make in traversing $\frac{1}{4}$ of a mile?

39. *a.* Construct an equilateral triangle and prove that each angle is equal, being 60°.

b. Construct a right triangle having an acute angle of 30°.

40. A circular fish pond which covers an area of 5 acres and 100 square rods is surrounded by a walk 5 yards wide. Compute the cost of graveling the walk at $6\frac{1}{4}$ cents per square yard.

41. A ladder 40 feet long is placed so as to reach a window 24 feet high upon one side of the street, and on turning the ladder over to the other side of the street it just reaches a window 32 feet high. What is the width of the street?

42. Show that if a line divides two sides of a triangle proportionally, the line is parallel to the third side of the triangle.

43. If $3 \times = AB$, form a proportion between the quantities 3, $\times$, A, and B. Prove your answer.

44. A building 90 feet high casts a shadow 45 feet long. Upon measurnig the shadow cast by a steeple at the same time, the length of the shadow was found to be 85 feet. How high was the steeple?

45. Plans for structural work called for an angle of 22° 30′. How would you construct the angle?

After completing the work, add and sign the following statement:
I hereby certify that the above work is entirely my own.

(Signed)

W7-DGQ-524